What do financial markets say about the exchange rate?*

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Preliminary and incomplete, comments welcome!

Abstract

We study a broad set of financial market structures disciplined by the absence of international arbitrage opportunities. We derive a general set of restrictions on the exchange rate imposed by these structures without taking a stance on the specifics of economic environments. We investigate which types of departures from complete markets could be helpful in resolving classic currency puzzles. In circumstances when financial markets are informative about the exchange rate, the constraints they impose on its behavior yield counterfactual implications. In contrast, financial market structures that are consistent with observed empirical properties of the exchange rate impose few constraints on its equilibrium behavior.

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Introduction

The exchange rate (FX) macro disconnect puzzle is a broad set of evidence suggesting little to no relationship between exchange rates and the macroeconomy (Obstfeld and Rogoff, 2001). As an alternative, some macroeconomic models of exchange rate point towards the financial market as the source of exchange rate volatility. Indeed, an immediate role of the exchange rate is to balance out demand and supply in the currency market. To the extent the currency market is linked to other financial markets, a natural question is how much information do financial markets contain about the exchange rate. Specifically, suppose one adopts lack of international arbitrage opportunities as a disciplining principle. How much is the possible equilibrium behavior of the exchange rate constrained by the local Euler equations, that is correct valuation of locally traded assets with local stochastic discount factors (SDF)?

Interestingly, one may have different priors in this regard. On the one hand, because the exchange rate is a separate asset class, knowing prices and returns of locally traded assets might contain little information about the exchange rate. After all, few assets — typically derivatives — can be priced directly by no arbitrage. On the other hand, the exchange rate is a special asset which converts one unit of account into another unit of account, and local financial markets contain information about the respective units of account. To make this concrete, consider a special case of complete local financial markets that provide information about state prices in local currency (i.e., the £ price of a state of the world in the UK and its $ price in the US). If any agent (e.g., an intermediary) has access to both local financial markets, then there exists a unique value of the exchange rate depreciation which is consistent with no arbitrage. In this sense, the exchange rate is fully pinned down by the information from local financial markets, resulting in the asset market view (AMV) of the exchange rate:

$$\Delta s_{t+1} = m^*_{t+1} - m_{t+1}, \tag{1}$$

where Δs is the log home currency depreciation rate, m (m^*) is the log SDF of the home (foreign) investor (Backus, Foresi, and Telmer, 2001).

Of course, the case of complete asset markets, even if an important benchmark, is very special and likely unrealistic. Therefore, we ask whether this logic that links the exchange rate to equilibrium properties in local financial markets extends beyond complete markets to a general structure of local and global financial markets.[1] In moving away from complete markets, one can relax which assets are traded (e.g., only some bonds and/or some stocks) or who trades (e.g., situations of imperfect integration or in which households trade through intermediaries). All these possible departures from complete markets would affect equilibrium properties of the exchange rate. This range of possible environments poses a conceptual challenge to drawing general lessons about the effect of market incompleteness, as every such departure from complete markets seemingly requires a case-by-case analysis.

This paper shows that there exists a general equilibrium relationship between the exchange rate and local financial markets that must hold independently of the degree of asset market completeness and the structure of traded risks. This relationship is

[1]Note that in the example above complete markets were not essential, as the pricing of the exchange rate relied on the presence of an Arrow security for a given state of the world in each of the two local markets connected via an intermediary. In this sense, pinning down the state-specific exchange rate depreciation requires neither a full set of AD securities, nor an integrated market. Furthermore, one does not need to know any SDFs, the knowledge of the £ and $ prices of the Arrow security is sufficient; a relatively expensive £ price requires a $ depreciation.

portable across all models — general or partial equilibrium (i.e., any model of SDF and returns or any macro model with a financial market) — that respect no arbitrage in financial pricing. Thus, it could be applied in any environment without fully solving for equilibrium, which is often difficult analytically and even computationally.

Away from complete markets, these constraints do not uniquely pin down the exchange rate, yet often impose interesting restrictions on its properties. While under complete markets the innovations to the exchange rate coincide with the innovations to the SDF differential, in the general case only their *contemporanous* projections on globally traded risks ϵ^g — risks that both home and foreign investors can trade — line up:[2]

$$\text{proj}(\,\widetilde{\Delta s}_{t+1}\,|\epsilon^g_{t+1}) = \text{proj}(\,\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}\,|\epsilon^g_{t+1}), \tag{2}$$

where $\widetilde{x}_{t+1} \equiv x_{t+1} - E_t x_{t+1}$ is an innovation to the variable x.

When globally traded risks span both the FX and the SDF differential, then both sides of Equation (2) are equal to innovations of the exchange rate and the SDF differential, respectively, as in the complete-markets case. Another special case of incomplete markets that is extensively discussed in the literature is when markets are fully integrated — both investors have access to the same assets, including risk-free assets in both currencies — (e.g., Lustig and Verdelhan, 2019, Maurer and Tran, 2021, Sandulescu, Trojani, and Vedolin, 2020). In that case the exchange rate itself is a global risk because it can be replicated via the carry trade. Then the left-hand side of the Equation is simply the innovation in the depreciation rate and it is equal

[2]Formally, globally traded risks ϵ^g_{t+1} are all random variables that can be spanned by returns in each of the local markets in *local currencies*, extending the concept of locally traded Arrow securities discussed above.

to the projection of SDFs on the exchange rate. Our results apply beyond the case when the exchange rate and/or SDFs are spanned by globally traded risks, covering a wide range of additional market structures that impose less stringent constraints on the equilibrium relationship between the FX depreciation and SDF differential.

While our first result in (2) characterizes restrictions on exchange rate shocks (innovations), our second result concerns the expected depreciation, or FX risk premium. We show that the expected depreciation rate is similar to the one under complete markets when traded asset returns span the exchange rate; otherwise, it is unconstrained.[3] In the latter case, we use an additional "good deal" bound (exclusion of quasi-arbitrages) to limit the expected depreciation given the unspanned component of FX volatility. We further prove that constraints associated with the two results exhaust all possible restriction imposed by the financial markets on the exchange rate, i.e. they are necessary and sufficient to preclude international arbitrage.

We use these results to investigate which specific economic environments could be helpful in resolving the currency puzzles that arise in equilibrium models with complete markets. In this context we study what SDFs prescribed by equilibrium models imply about the exchange rate across a variety a market structures. Complete-markets-based models often yield counterfactual behavior of the volatility (Brandt, Cochrane, and Santa-Clara, 2006), cyclicality (Backus and Smith, 1993), and FX risk premium (Fama, 1984).

What emerges in our analysis as the organizing principle is not market completeness per se, but rather the prominence of globally traded risks. It is necessary to both

[3]More precisely, if FX is spanned only by globally traded risks, then FX risk premium is the same as under complete markets. When FX is spanned by globally and locally traded risks, then FX risk premium may feature an additional term that is still fully determined by the financial markets. Without spanning, FX risk premium is unconstrained by the financial market altogether.

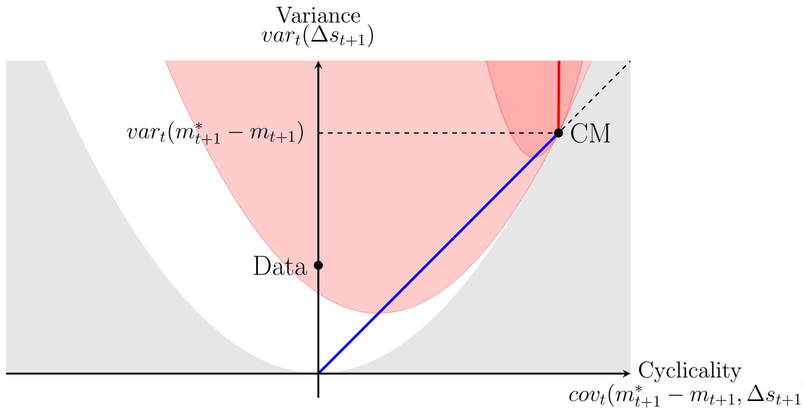
move away from integrated markets and have sparsely traded risks to loosen the grip of financial markets on the equilibrium exchange rate. When globally traded risks are sparse, local financial markets contain little information about possible equilibrium exchange rate behavior which is shaped by forces outside these markets. This feature is helpful in accommodating empirical evidence on the financial FX disconnect.

Figure 1 offers one way to visualize this insight. This figure considers the trade-off in capturing both variance $var_t(\Delta s_{t+1})$ and cyclicality $cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})$ of the exchange rate. The point labeled as 'Data' is a stylized representation of the evidence: a relatively stable and approximately acyclical exchange rate. The point labeled 'CM' represents the complete markets case where $var_t(\Delta s_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})$. The white cone (the entire area outside of grey) depicts all mathematically feasible combinations of the two objects. This area also corresponds to the case when the variance of the global risk component is equal to zero, i.e. $\text{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} \,|\epsilon^g_{t+1}) = 0$.

The pink cones in Figure 1 show the feasible combinations of FX volatility and cyclicality as the variance of the global shocks increases to 25% (the larger cone) and to 90% (the smaller cone), degenerating in the limit to the red vertical line (ray) when global shocks span 100% of the SDF differential. The blue 45-degree line segment from the origin to CM corresponds to the case when FX is spanned by global shocks, i.e. the FX is a global risk itself (e.g., the aforementioned case of integrated markets). Over this segment, $cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) = var_t(\Delta s_{t+1})$, as the SDF differential and the exchange rate must be perfectly correlated along global shocks.

It is clear from this picture that models with a dominant role of globally traded risks face similar challenges in terms of FX volatility and cyclicality as complete market

Figure 1: The impact of global shocks on variance and cyclicality of exchange rates



Note: The figure illustrates the trade-offs in matching volatility and cyclicality of the exchange rate (see text). The point labeled Data is a stylized representation of the evidence regarding the depreciation rates; the point labeled CM represents the complete market setting. The grey area represents infeasible combinations of volatility and cyclicality due to the Cauchy-Schwarz inequality.

models. In contrast, models that imply relatively small role for global shocks have a potential for capturing the evidence. We apply our theoretical results to the specific economic settings to see what kind of variance-cyclicality profiles could be obtained. We find that if only one dimension of market incompleteness is compromised — i.e., either market integration or the set of traded risks — the connection between the exchange rate and SDFs is similar to that of complete markets, resulting in the same puzzles.

Allowing for both types of market incompleteness at the same time — namely, intermediated markets with sparsely traded risks — results in limited restrictions on the exchange rate, i.e. local financial markets do not say much about it, and that leads to a relaxation of the tension between the volatility and cyclicality puzzles. As a result, such a framework has a potential to capture the puzzling features of the data. We conclude that intermediated markets in which the nature of economic shocks is varied and not too strongly related across countries are particularly promising in addressing the puzzles quantitatively.

As a last element of our analysis we develop evidence regarding the shock structure. We consider G-10 countries, U.S. vs foreign on a bilateral basis, from 1988 to 2022 at a monthly frequency. We designate sovereign bonds of maturities ranging from 2 to 10 years, and various stock indexes (the market, value-growth, and industry portfolios) as risky assets. First, we evaluate whether the depreciation rate can be spanned by asset returns. The answer is no: the largest spanning regression R^2 is 45% for Canada (vs the U.S.), the lowest is 25% for Switzerland. Thus, unspanned shocks play an important role in the variation of the exchange rate. This conclusion is consistent with that of Chernov and Creal (2023). The results depend on the choice of spanning assets. It is an interesting topic for future research to establish the optimal currency spanning portfolio.

Next, we quantify global shocks in this empirical setting using two methods. First, we use canonical correlation analysis to find maximally correlated portfolios in a pair of countries. Ideally, the maximum correlation should equal to 1, but we allow for correlations as low as 0.6 for the portfolios to qualify as measures of global shocks. Second, we use shocks that are commonly used as global in the literature: the Volatility Index (VIX), the Global Financial Cycle (GFC, Miranda-Agrippino and Rey, 2020), and the Excess Bond Premium (EBP, Gilchrist and Zakrajsek, 2012). This method assumes that one can find shock-replicating portfolios in each country. Regardless of the method, global shocks contribute little to the variation in exchange rates: most countries have no more than 10% of FX variation explained by global shocks. Thus, the evidence is supportive of our theoretical conclusions that currency puzzles should be resolved with models allowing for plenty of unspanned and local shocks to the exchange rate.

Related literature. We derive general restrictions on the exchange rate given financial market structure in each of the two countries. We apply these restrictions to the famous facts about exchange rates such as their relatively low volatility and weak relation to business cycles. The resulting implication that exchange rates features large unspanned and local shocks is very much in the spirit of Hansen and Jagannathan (1991) agnostic characterization of asset-pricing models. Departures from complete markets in the context of currency puzzles is explored by Lustig and Verdelhan (2019). They consider a special case where the exchange rate is spanned because each country's investor can trade the other country's risk-free bond. That makes it difficult to capture volatility and cyclicality puzzles jointly. Jiang, Krishnamurthy, Lustig, and Sun (2022) consider a similar incomplete-market setting with international access to trading in risk-free bonds but complemented by safe asset demand for dollar bonds. This feature leads to wedges in the Euler equations, which are ruled out in our setting. One implication of these wedges is that the exchange rate is affected by the convenience yield in addition to risks spanned by the SDFs. Chernov and Creal (2023) emphasize inability of bonds to span exchange rates and propose an affine term structure model with martingale shocks to the SDF, which affect the exchange rate but not bond prices. RBC models of exchange rates are represented by Verdelhan (2010) (habits), Colacito and Croce (2011) (long-run risk), and Farhi and Gabaix (2016) (disasters), among many others. Exchange rate models with intermediation include Gabaix and Maggiori (2015), Gourinchas, Ray, and Vayanos (2022), and Itskhoki and Mukhin (2021).

1 Setup

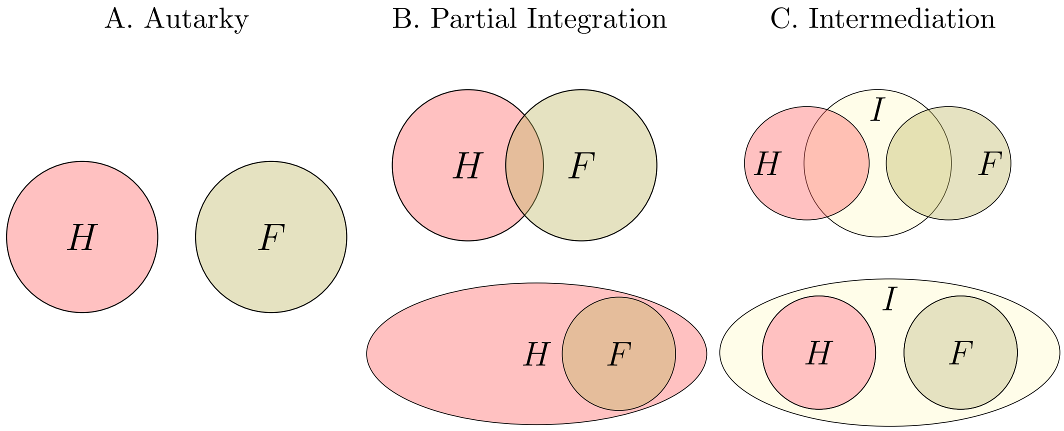
We are interested in restrictions on the behavior of exchange rates coming from properties of other asset returns. To answer this question, we introduce a general framework and derive two sets of restrictions — on the risk (innovation) of the exchange rate depreciation and the expected exchange rate depreciation. In broad strokes, we fix the financial market structure of domestic and foreign households, that is, the returns on assets these households can trade and the SDFs of the households. Next, we establish which restrictions this financial structure imposes on the exchange rate between the two countries.

1.1 Market structure

We consider settings with two representative households, h for home, and f for foreign. Each household can trade a set of assets, H and F, respectively. Those sets can contain subsets of local assets and foreign assets converted to local currency. Figure 2 demonstrates some examples. For instance, in autarky H contains domestic stocks and bonds, while F contains the foreign ones. When markets are integrated, H and F contain identical assets but expressed in respective currencies, e.g., H may include a domestic sovereign bond and a foreign equity index converted to domestic currency, while F contains domestic bond converted to foreign currency and foreign equity index. If markets are complete, H and F contain the full set of Arrow-Debreu securities expressed in respective currencies. Market completeness is a particular case of full market integration where securities span all possible risks.

Further, we consider a set I of assets traded by an international arbitrageur. Assets

Figure 2: Examples of Market Structures



The figure illustrates different market structures. H and F are the set of assets invested in by the home and foreign household. Panel A corresponds to financial autarky. Panel B corresponds to partial integration, symmetric or asymmetric. Panel C corresponds to an intermediated market, with an intermediary I trading some or all assets.

can be included in this set for two reasons. First, it could be that home and foreign households trade some assets in common, as in the partially integrated cases above. Then, either h or f can be considered the international arbitrageur, with $I = H$ or $I = F$, respectively. Second, it could be that a financial intermediary trades across borders even if households do not, as in the examples in panel C of Figure 2. In this case, I are the assets from H and F that the intermediary can trade. We require that H and F each contain a risk-free bond in the respective currency, and I contains both risk-free bonds.

Our main result is that restrictions on the exchange rate in this large family of market structures are detemined by the properties of returns in $H \cap I$ expressed in domestic currency and returns in $F \cap I$ expressed in foreign currency. To continue our examples, if markets are partially integrated and $I = H$, then $H \cap I = H$ are the assets traded by the domestic household, $F \cap I = F \cap H$ are the assets traded by both households. In intermediated markets, $H \cap I$ is the set of assets traded both by the domestic household and the intermediaries; ditto for $F \cap I$.

The base assets in the set $H \cap I$ have log returns $\boldsymbol{r}_{t+1} = (r_{1,t+1}, \ldots, r_{N,t+1})$. We assume this collection includes a risk-free asset with return r_{ft} in home currency known at time t. The corresponding set of all feasible portfolio returns is $\boldsymbol{r}_{p,t+1} = \{r_{p,t+1} | \exists \boldsymbol{w}_t \in \mathbb{R}^N : \boldsymbol{w}_t' \boldsymbol{\iota} = 1, r_{p,t+1} = \log(\boldsymbol{w}_t' \exp(\boldsymbol{r}_{t+1}))\}$. Furthermore, we assume that asset returns are log-normal, that is $\boldsymbol{r}_{t+1}$ are multivariate normal, $MVN(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$. Similarly, the returns of base assets in $F \cap I$ are $\boldsymbol{r}^*_{t+1}$ in foreign currency, log-normal of size N^*, and contain a foreign-currency risk-free rate of r^*_{ft} . The corresponding set of portfolio returns is $\boldsymbol{r}^*_{p,t+1}$. Throughout the paper, we use the Campbell and Viceira (2002) approximation for log portfolio excess returns in the relevant derivations as described in the Appendix.

1.2 Pricing Assumptions

Local investors. We specify valuation mechanisms by each representative household with a given SDF m at home and m^* abroad. These SDFs value assets as follows.

Assumption 1. *The domestic (log) stochastic discount factor m_{t+1} prices all assets in H in domestic currency. In particular, it satisfies the Euler equation:*

$$\forall r_{t+1} \in \boldsymbol{r}_{p,t+1} : \quad E_t \left[\exp(m_{t+1} + r_{t+1})\right] = 1. \tag{3}$$

Similarly, the foreign log SDF m^*_{t+1} prices all assets in F in foreign currency, and

$$\forall r^*_{t+1} \in \boldsymbol{r}^*_{p,t+1}: \quad E_t\left[\exp(m^*_{t+1} + r^*_{t+1})\right] = 1. \tag{4}$$

Recall that $\boldsymbol{r}_{p,t+1}$ ($\boldsymbol{r}^*_{p,t+1}$) is the set of feasible portfolio returns constructed from assets in $H \cap I$ ($F \cap I$). Thus (3) and (4) require only pricing of assets in sets $H \cap I$ and $F \cap I$, respectively. These Euler equations are all that is needed for our formal results. Nevertheless, in many economic environments it is reasonable to assume that the same home and foreign SDFs price all assets in H and F, respectively. Assumption 1 can be viewed as the definition of local financial market equilibrium that we use in our analysis.[4]

We focus on situations with log-normal SDFs. The Euler equations imply that expected excess returns are proportional to the covariance with the stochastic discount factors. In our log-normal setting, this corresponds to:

$$\forall r_{t+1} \in \boldsymbol{r}_{p,t+1}: \quad E_t(r_{t+1}) + \frac{1}{2}var_t(r_{t+1}) = r_{ft} - cov_t(m_{t+1}, r_{t+1}), \tag{5}$$

$$\forall r^*_{t+1} \in \boldsymbol{r}^*_{p,t+1}: \quad E_t(r^*_{t+1}) + \frac{1}{2}var_t(r^*_{t+1}) = r^*_{ft} - cov_t(m^*_{t+1}, r^*_{t+1}). \tag{6}$$

We do not take a stand on the origins of these discount factors as long as the combination of SDFs and returns satisfy Assumption 1. In some applications, the discount factors represent optimal decisions of domestic and foreign households. For exam-

[4]Note that equilibrium in the financial market may involve borrowing or short-sale constraints, infrequent portfolio adjustment, or convenience yield on certain assets. In all such cases, some Euler equations do not always hold with equality, and in our analysis this simply requires redefining sets H and F to exclude such assets (for a given time period t). In this case, conditions (3) and (4) can be thought of as definitions of sets H and F rather than an assumption.

ple, with CRRA utility, $m_{t+1} = -\gamma \Delta c_{t+1}$ where c_t being log aggregate domestic consumption and γ a coefficient of risk aversion. Further, c_t could be treated as observable object that does not change with economic environment, or could be allowed to change endogenously. In other applications, the discount factors are simply a representation of the risk-return relation among assets traded in a country. For example, the SDF could be constructed from asset returns as $m_{t+1} = \boldsymbol{\lambda}_t' \boldsymbol{r}_{t+1}$, with $\boldsymbol{\lambda}_t \in \mathbb{R}^N$, in the spirit of Hansen and Jagannathan (1991).[5] Here, the SDF is changing with the market structure that determines the subsets of assets traded by investors.

Our results lead to a set of cross-equation restrictions on the joint behavior of endogenous variables: returns, SDFs, and the exchange rate. These relationships must hold in *any* equilibrium environment that respects no arbitrage. Further, within an equilibrium, there often exist many SDFs that satisfy Assumption 1; our results hold for every admissible SDFs.

International arbitrage. Our assumptions so far ensure that each of the domestic and foreign sets of asset returns have standard and tractable properties. Importantly, note that none of them involve explicitly the exchange rate. We now turn to the connection between domestic and foreign asset returns. Specifically, we assume that there are no arbitrage opportunities for assets in I, i.e. assets traded by an international investor. This implies existence of an SDF for the international arbitrageur. Unlike for households, we do not assume any knowledge of this SDF beyond its existence.

[5] More precisely, this expression could be viewed as a log-approximation to SDFs which are constructed from portfolio returns. We can write $m_{t+1} = \log\left(\boldsymbol{\lambda}_t' \exp(\boldsymbol{r}_{t+1})\right)$, the same way we defined $\boldsymbol{r}_{p,t+1}$ above. One can then consider, for example, weights $\boldsymbol{\lambda}_t$ that minimize the variance or the entropy of the SDF.

The set of returns in I combines the domestic and foreign set of international returns converted to the domestic currency.[6] Following our notations, international portfolios are generated by the base assets $\boldsymbol{r}^I_{t+1} = (\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1} + \Delta s_{t+1})$, where Δs_{t+1} is the log home currency depreciation rate. We denote the set of international portfolios generated by these base assets by $\boldsymbol{r}^I_{p,t+1}$.

Assumption 2. *There are no arbitrage opportunities in the set of international returns $\boldsymbol{r}^I_{p,t+1}$, that is:*

$$\forall r_{p,t+1} \in \boldsymbol{r}^I_{p,t+1} : \; var_t(r_{p,t+1}) = 0 \quad \Rightarrow \quad E_t(r_{p,t+1}) = r_{ft}. \tag{7}$$

In words, any portfolio that has no risk must earn the risk-free rate of return.[7]

1.3 Global, local and unspanned shocks

Intuitively, returns are affected by a collection of shocks, some of which are local to each economy, $\boldsymbol{\epsilon}_{t+1}$ or $\boldsymbol{\epsilon}^*_{t+1}$, while others are common to both, i.e. global shocks $\boldsymbol{\epsilon}^g_{t+1}$. We denote with tilde the innovation (or shock) for any variable x, that is $\widetilde{x}_{t+1} \equiv x_{t+1} - E_t x_{t+1}$.[8] With this, we define the set of globally-traded shocks, or global shocks for short.

Definition 1. *The set of global shocks is* $\boldsymbol{\epsilon}^g_{t+1} = \{\epsilon^g_{t+1} | \exists \boldsymbol{\lambda} \in \mathbb{R}^N, \boldsymbol{\lambda}^* \in \mathbb{R}^{N^*} : \epsilon^g_{t+1} = \boldsymbol{\lambda}' \widetilde{\boldsymbol{r}}_{t+1} = \boldsymbol{\lambda}^{*\prime} \widetilde{\boldsymbol{r}}^*_{t+1}\}$.

[6]Our conclusions are unchanged if we focus on international arbitrage in foreign currency.

[7]In our log-normal setting, condition (7) is equivalent to the absence of arbitrage opportunities. In more general settings, it is a necessary condition for no arbitrage.

[8]Note that $var_t(\widetilde{x}_{t+1}) = var_t(x_{t+1})$ and we use this notation interchangeably.

Thus, global shocks can be traded by local investors in their local currency in both countries. Appendix A shows how to construct a basis of this space from the covariance matrix of $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$. Local shocks $\boldsymbol{\epsilon}_{t+1}$ and $\boldsymbol{\epsilon}^*_{t+1}$ are the residuals of return innovations, $\widetilde{\boldsymbol{r}}_{t+1}$ and $\widetilde{\boldsymbol{r}}^*_{t+1}$, after controlling for global shocks.

Global shocks can arise because of common underlying economic shocks (e.g., productivity) that determine returns in both countries. Alternatively, global shocks can emerge without common fundamental shocks as a result of asset trading across countries — either directly by households or via an intermediary.

As an example, consider partially integrated markets such as the ones in Figure 2B. Imagine that $\boldsymbol{r}_{t+1} = (r_{ft}, r_{1,t+1}, r_{2,t+1}, r^*_{ft} + \Delta s_{t+1}, r^*_{1,t+1} + \Delta s_{t+1})$ and $\boldsymbol{r}^*_{t+1} = (r^*_{ft}, r^*_{1,t+1}, r^*_{2,t+1}, r_{ft} - \Delta s_{t+1}, r_{1,t+1} - \Delta s_{t+1})$. In such a setting, the domestic investor can construct a portfolio with a return $r^*_{1,t+1} - r^*_{ft}$ by buying the foreign risky asset 1 and by selling the foreign risk-free asset, both converted into domestic currency. Similarly, the foreign investor can construct a portfolio with a return $r_{1,t+1} - r_{ft}$. As a result, both $\widetilde{r}_{1,t+1}$ and $\widetilde{r}^*_{1,t+1}$ are in the set of global shocks $\boldsymbol{\epsilon}^g_{t+1}$.[9] Furthermore, here the FX risk $\widetilde{\Delta s}_{t+1}$ is also a global shock: it can be traded by both households through their respective carry trades.

Finally, we refer to any other sources of variation orthogonal to asset returns $(\widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1})$, or equivalently orthogonal to local and global shocks $(\boldsymbol{\epsilon}^g_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1})$, as unspanned shocks.

[9] A practical example of such global shocks arises in the context of commodity (e.g., oil) futures denominated in different currencies, or stocks of the same company traded in jurisdictions with different currencies (e.g., Royal Dutch Shell).

Exchange rate depreciation. We can use this taxonomy of shocks to uniquely decompose the innovation to the depreciation rate as follows:

$$\widetilde{\Delta s}_{t+1} = g_{t+1} + \ell_{t+1} + u_{t+1}, \tag{8}$$

where g_{t+1} is a linear combination of global shocks $\boldsymbol{\epsilon}^g_{t+1}$, ℓ_{t+1} is a linear combination of both types of local shocks, $\boldsymbol{\epsilon}_{t+1}$ and $\boldsymbol{\epsilon}^*_{t+1}$, and u_{t+1} is unspanned.

Thus, there are four components to the exchange rate depreciation, Δs_{t+1}. The first is the conditional expectation $E_t \Delta s_{t+1}$. Then, there are two types of shocks spanned by assets, a global component g_{t+1} and a local component ℓ_{t+1}. Finally, there can be unspanned shocks u_{t+1}. This decomposition will play a central role in our characterization of restrictions on the behavior of the exchange rate.

Relatedly, one can construct the spanned components directly from asset returns:

$$\widetilde{\Delta s}_{t+1} = \widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1} + u_{t+1}, \tag{9}$$

with $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ two portfolios with best R^2 for explaining the exchange rate.[10] Mechanically, the residual coincides with the unspanned component u_{t+1} in equation (8). If this unspanned component is equal to 0, the depreciation rate is spanned by asset returns, and the difference between the shocks to returns on the two portfolios replicates the exchange rate shock exactly.

[10]Formally, the portfolios maximize $R^2 = 1 - var_t(\Delta s_{t+1} - (\widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1}))/var_t(\Delta s_{t+1})$.This pair of portfolios is not unique when global shocks are present. All of our results hold for any such pair.

2 The general asset market view of exchange rates

In this section, we characterize the restrictions on the behavior of the exchange rate imposed by the absence of international arbitrage and given the properties of returns on traded assets, $\boldsymbol{r}$ and $\boldsymbol{r}^*$, and local SDFs m and m^* that price them. We show that Assumptions 1 and 2 impose two sets of necessary restrictions on the depreciation rate — one on the shocks to the depreciation rate $\widetilde{\Delta s}_{t+1}$ and another on the expected depreciation rate $E_t \Delta s_{t+1}$.

We demonstrate that in a complete market setting these two sets of restrictions lead to the well-known asset market view of exchange rates and the puzzles that come with it. Subsequent analysis spells out the implications of these restrictions for a much larger set of market structures and revisits the puzzles in light of these results. All the proofs are in Appendix B. Appendix C proves the sufficiency of our key results: if the two sets of restrictions hold, Assumption 2 about the absence of international arbitrage opportunities is valid.

2.1 Exchange rate shocks

We show that the component of the depreciation rate that loads on global shocks, g_{t+1}, must coincide with the component of the difference of SDFs that loads on global shocks.

Proposition 1. *Under Assumptions 1 and 2,*

$$proj(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}) = proj(\widetilde{\Delta s}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}) = g_{t+1}. \tag{10}$$

Said differently, start from any pair of admissible local SDFs and regress them on all global shocks. The predicted value of this regression is equal to the global component of the exchange rate, g_{t+1}:

$$m^*_{t+1} - m_{t+1} = g_{t+1} + v_{t+1} \qquad \text{with} \quad v_{t+1} \perp \boldsymbol{\epsilon}^g_{t+1}. \tag{11}$$

What is missing in Proposition 1 is just as important as what is there. Local financial markets do not impose any restrictions on the component of the depreciation rate loading on either local shocks $(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1})$ or its unspanned component u_{t+1}. Thus, in general, financial markets impose less restrictions on the exchange rate as compared with complete markets.

How does the absence of arbitrage lead to this result? In complete markets, local and foreign investors must agree on the price of all payoffs after conversion to a common currency: $cov_t(m_{t+1}, r_{t+1}) = cov_t(m^*_{t+1} - \Delta s_{t+1}, r_{t+1})$ for every r_{t+1}. Proposition 1 comes from a generalization of this result. To preclude arbitrage opportunities, local and foreign investors must only agree on the price of risks that they both trade — the global shocks.

Without a change of currency, the argument is standard: the international arbitrageur can buy the global shock ϵ^g_{t+1} in the home market and sell it in the foreign market.[11] Because this portfolio is riskless, the two risk premia must coincide, $cov_t(m_{t+1}, \epsilon^g_{t+1}) = cov_t(m^*_{t+1}, \epsilon^g_{t+1})$. In Appendix B.2, we show that this logic extends to the case with currency conversion, and no arbitrage requires an adjustment to expected returns of $cov_t(\Delta s_{t+1}, \epsilon^g_{t+1})$, the so-called quanto adjustment. This implies that the comovement of the depreciation rate with global shocks must be the same as

[11] See, for example, Chen and Knez (1995).

that of the relative SDFs, $cov_t(m^*_{t+1} - m_{t+1}, \epsilon^g_{t+1}) = cov_t(\Delta s_{t+1}, \epsilon^g_{t+1})$. Conversely, for shocks that are not traded by both investors, it is impossible to construct candidate arbitrage portfolios that relate pricing in the two markets (see Appendix C).

2.2 Expected depreciation rate

We turn to restrictions on the behavior of the expected depreciation rate. These restrictions depend on the relation of the exchange rate with asset returns. Start from the projection of the exchange rate on asset returns, represented by two portfolio $r_{p,t+1}$ and $r^*_{p,t+1}$ as in equation (9). Recall that when $r_{p,t+1}$ and $r^*_{p,t+1}$ span the exchange rate, the unspanned component u_{t+1} is equal to 0. We define δ_t as the difference of the two portfolios' expected returns:

$$\delta_t \equiv \left[r_{ft} - cov_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2}var_t(r_{p,t+1})\right] - \left[r^*_{ft} - cov_t(m^*_{t+1}, r^*_{p,t+1}) - \frac{1}{2}var_t(r^*_{p,t+1})\right]. \tag{12}$$

The following proposition relates the behavior of the expected depreciation rate to spanning of the exchange rate and this quantity, which only depends on asset returns and local SDFs.

Proposition 2. *The expected depreciation rate is pinned down if and only if the exchange rate is spanned by asset returns, that is when $u_{t+1} = 0$. In this case, it is:*

$$E_t \Delta s_{t+1} = \delta_t = \underbrace{r_{ft} - r^*_{ft}}_{UIP} - \underbrace{cov_t(m_{t+1}, \Delta s_{t+1})}_{Exchange\ rate\ risk\ premium} - \underbrace{\frac{1}{2}var_t(\Delta s_{t+1})}_{convexity} + \theta_t, \tag{13}$$

where $\theta_t = cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{p,t+1})$*. This quantity collapses to* $\theta_t = 0$ *when the exchange rate is spanned by global shocks.*

The most important implication of Proposition 2 is that it delineates two cases: either local market pricing determines expected depreciation exactly, or it says nothing about it. The expected depreciation rate is closely related to the risk premium for exchange rate risk. Exposure to this risk can be obtained by engaging in the carry trade. This risk premium is pinned down by pricing in local financial markets only if the international arbitrageur can use locally traded assets to perfectly offset this risk. Therefore, the absence of arbitrage has no bearing on this quantity if the exchange rate is not spanned by asset returns, that is, $u_{t+1} \neq 0$.

Spanned exchange rate. When the exchange rate is spanned, the international arbitrageur uses the two local markets to price the exchange rate risk. Hence, the two local SDFs play a role in the expected depreciation rate. This insight explains the presence of the novel adjustment term θ_t in equation (13) relative to the standard complete market formula (with $\theta_t = 0$). It also leads to a symmetric expression to equation (13) which emphasizes the foreign SDF m^*_{t+1}:

$$\delta_t = r_{ft} - r^*_{ft} - cov_t(m^*_{t+1}, \Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) + \theta^*_t, \tag{14}$$

with $\theta^*_t = cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{p,t+1})$.

It is only when the local investors are able to replicate the exchange rate on their own that their individual Euler equations are enough to obtain the expected depreciation. If the home (foreign) investor can trade both spanning portfolios, then $\theta_t = 0$ ($\theta^*_t =$

0), and the standard complete market formula for the home (foreign) investor holds. For example, this situation occurs in settings in which the home investor acts as an international arbitrageur. For both home and foreign investors to price the exchange rate risk, they must be able to trade it, that is, the exchange rate is a global shock.

Unspanned exchange rate. When the exchange rate is not spanned by traded assets, its expectation can deviate from this formula by an arbitrary wedge,

$$E_t \Delta s_{t+1} = \delta_t + \psi_t. \tag{15}$$

This complete flexibility might lead to implausibly large trading profits for the international investor. One can be more informative about these deviations ψ_t by imposing a condition that is stronger than the absence of arbitrage (Assumption 2).

Assumption 3. *(No quasi-arbitrage) There is an upper bound B on Sharpe ratios in international markets:*

$$\forall r^I_{p,t+1} \in \boldsymbol{r}^I_{p,t+1} : \quad \left| E_t(r^I_{p,t+1}) + \frac{1}{2} var_t(r^I_{p,t+1}) - r_{ft} \right| \leq B \sqrt{var_t(r^I_{p,t+1})}. \tag{16}$$

This assumption restricts the Sharpe ratio of trades in international markets. Such bounds have a long tradition in finance, going back to Cochrane and Saa-Requejo (2000), Kozak, Nagel, and Santosh (2020), and Ross (1976). Intuitively, it can be motivated by the view that if trades that are too profitable emerged in equilibrium, new financial institutions would step in to take advantage of them. Under this view we obtain the following condition.

Proposition 3. *Under Assumption 3, the wedge ψ_t in the expected depreciation rate* *must satisfy:*

$$\left|\psi_t + \frac{1}{2}var_t(u_{t+1})\right| \leq B\sqrt{var_t(u_{t+1})} \equiv B\sqrt{(1-R^2)var_t(\Delta s_{t+1})}, \tag{17}$$

where R^2 is the R-squared in the regression of Δs_{t+1} on $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$.

This proposition limits possible expected depreciations in the case of an unspanned exchange rate. It indicates that deviations from the risk premium in the spanned case are bounded by the volatility of unspanned shocks.

3 Implications for the currency puzzles

In this section we discuss how the different market structures are capable of speaking to the key currency puzzles: cyclicality, volatility, and forward premium. Thus the thought experiment in this section is that the SDFs are emerging from economic theory, i.e., they are IMRS of households in each country, and a researcher is trying to characterize the resulting exchange rate.

3.1 Complete markets

The complete markets case is the relevant benchmark for our discussion as it forms the backbone of many models attempting to explain the currency puzzles. Financial markets are complete when investors have access to the full set of Arrow-Debreu securities in both markets. In this setting, $\boldsymbol{\epsilon}^{g}_{t+1}$ spans all possible risks. Then Proposition 1 implies

$$\widetilde{m}^{*}_{t+1} - \widetilde{m}_{t+1} = \widetilde{\Delta s}_{t+1}. \tag{18}$$

Innovations to the depreciation rate must equal innovations to the difference of stochastic discount factors, completely pinning down exchange rate shocks.

This result leads to two puzzles about the behavior of the exchange rate. First, consider the variance of the depreciation rate:

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m^{*}_{t+1} - m_{t+1}) \\ &= var_t(m^{*}_{t+1}) + var_t(m_{t+1}) - 2cov_t(m_{t+1}, m^{*}_{t+1}). \end{aligned} \tag{19}$$

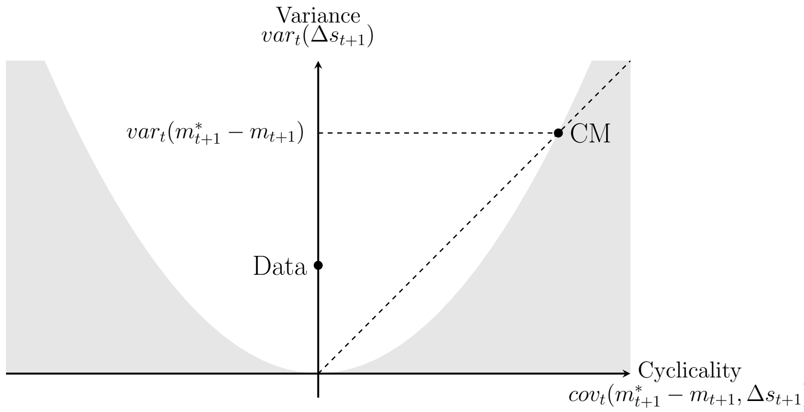
Brandt, Cochrane, and Santa-Clara (2006) argue that this equation creates a volatility puzzle, with the exchange rate being not volatile enough. Typically observed Sharpe ratios on domestic assets imply highly volatile SDFs, much more so than exchange rate depreciation. The mild correlation of macro quantities across countries suggests that the SDFs are not correlated enough for the last term of equation (19) to offset this high variance and obtain realistic exchange rate risk.

Further, the result (18) implies

$$var_t(\Delta s_{t+1}) = cov_t(\Delta s_{t+1}, m^{*}_{t+1} - m_{t+1}), \tag{20}$$

and $corr_t(\Delta s_{t+1}, m^{*}_{t+1} - m_{t+1}) = 1$. Changes in exchange rates must be perfectly correlated with relative marginal utilities of the domestic and foreign households, that is, the home currency depreciates in relatively good times for home investors.

Figure 3: Proposition 1 in complete markets



The figure illustrates implications of the complete market setting, labeled as CM, for the properties of depreciation rates. The point labeled Data is a stylized representation of the evidence regarding the depreciation rates. The grey area represents the infeasible combinations of volatility and criticality of depreciation rates due to the Cauchy-Schwarz inequality.

As pointed out by Backus and Smith (1993), this implication is counterfactual for various measures of good times leading to the cyclicality puzzle.

We introduce a visualization of these puzzles which we will revisit for other market structures. Figure 3 demonstrates the tension in capturing volatility, on the vertical axis, and cyclicality, on the horizontal axis, at once. The point labeled 'Data' is a stylized representation of the empirically observed variance of depreciation rates and basically absent cyclicality. Equation (20) implies that the complete markets case should be on the 45° line. We select a point on the vertical axis that is equal to $var_t(m^*_{t+1} - m_{t+1})$ and, according to the volatility puzzle, is higher than $var_t(\Delta s_{t+1})$ that we see in the data.[12] The point labeled 'CM' shows what the complete market setting implies. The distance between Data and CM is the essence of the volatility

[12]The Cauchy-Schwarz inequality implies that

$$cov_t^2(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) \leq var_t(\Delta s_{t+1}) \cdot var_t(m^*_{t+1} - m_{t+1}).$$

and cyclicality puzzles in complete markets.

Moving onto the forward premium puzzle, consider the application of Proposition 2. In the complete-market setting $\widetilde{\boldsymbol{r}}_{t+1}$ and $\widetilde{\boldsymbol{r}}^*_{t+1}$ span $\widetilde{\Delta s}_{t+1}$, and $\psi_t = 0$. The resulting expected depreciation rate leads to the currency risk premium puzzle. The risk premium for currency does generate deviations from uncovered interest parity (UIP). Standard international models struggle with generating the empirically observed magnitude of currency risk premium simultaneously with addressing the first two puzzles.

Using the Euler equations, we can express $r_{ft} - r^*_{ft}$ in equation (13) in terms of SDFs. As a result,

$$E_t \Delta s_{t+1} = E_t(m_{t+1} - m^*_{t+1}).$$

The mean depreciation rate must equal the mean of the difference of stochastic discount factors. Combining this equation with equation (18) we obtain the classic "asset market view" result for exchange rates

$$m^*_{t+1} - m_{t+1} = \Delta s_{t+1}, \tag{21}$$

which completely pins down the depreciation rate.

Interestingly, our derivation highlights that this result does not hinge on the classic notion of market completeness. It requires neither integration of markets nor spanning of all states of the world. Indeed, it is enough to be able to span $\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}$ and $\widetilde{\Delta s}_{t+1}$ in each country in order to apply Propositions 1 and 2 and obtain the

We fix $var_t(m^*_{t+1} - m_{t+1})$ at the value indicated on the vertical axis. This leads us to a space of mathematically feasible combinations of volatility and cyclicality. The grey area on the chart indicates the infeasible combinations.

complete-market result of equation (21). Such situations can arise in two cases.

First, the set of assets in each country is dense enough for the required spanning to hold. We can think of this situation as a limiting case of projecting $\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}$ and $\widetilde{\Delta s}_{t+1}$ on more and more rich set of assets until the R^2 of the projections converge to 1.

Second, the set of shocks in the economy that drive $m^*_{t+1} - m_{t+1}$ and Δs_{t+1} is sparse enough that there exist assets in both countries that allow to trade both the exchange rate and the SDFs, even is the set of assets is not very dense. This situation may occur in models where all equilibrium objects are driven by a few global macro shocks $\boldsymbol{\epsilon}^g$, such as productivity or monetary policy.

3.2 The path towards resolving the complete-market puzzles

While the literature typically focuses on the three puzzles simultaneously, our Propositions suggest that forward premium puzzle, i.e., an empirical measure of the expected depreciation rate, is affected by different features of the financial markets as compared to the cyclicality and volatility puzzles, which are driven by the properties of the exchange rate innovations.

Proposition 2 offers a simple path towards resolving the forward puzzle. As long as the underlying economic structure is such that the traded assets cannot span the deprecation rate, there are no constraints whatsoever on what its expectation should be. That opens a window to generating a theoretical currency premium that would be consistent with the observed one.

Proposition 1 and its application to complete markets implies that in order to resolve the cyclicality and volatility puzzles, the share of global shocks in the economy must be less than 100%. In order to assess how small this contribution should be, we revisit the graphic representation of the two puzzles in Figure 3.

The following relation quantifies the trade-off between cyclicality and volatility

$$\overbrace{var_t(\Delta s_{t+1})}^{\text{volatility}} \geq var(g_{t+1}) + \frac{\left(\overbrace{cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})}^{\text{cyclicality}} - var(g_{t+1})\right)^2}{var_t(m^*_{t+1} - m_{t+1}) - var(g_{t+1})}, \tag{22}$$

which is visualized by the red cones on Figure 1.[13] According to Proposition 1, the minimum variance of the exchange rate is $var(g_{t+1})$; it is attained when the cyclicality has the same value, $cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) = var(g_{t+1})$. This point corresponds to the vertex of the parabola, which lies on the segment of the 45-degree line between the origin and the complete markets point. To reduce cyclicality of the exchange rate, one has to increase the variance of the exchange rate by adding non-global components, which can have arbitrary correlation with the SDFs. This is exactly what is captured by the inequality in equation (22), which corresponds to the space inside of the parabola in the figure. Hence, if one starts with a macro-finance model where the variance of the exchange rate is already too large relative to the data, improving on cyclicality would further worsen the volatility puzzle.

Across models, the smaller is the role of global shocks, the larger is the space of combination of volatility and cyclicality. Thus, in what follows, we discuss the ability of various deviations from the complete-market setting to generate the requisite amount of variation due to global shocks. Specifically, we consider various forms

[13]This relation is a consequence of the Cauchy-Schwartz inequality applied to the non-global components of the exchange rate and SDF differential, $(\ell_{t+1} + u_{t+1})$ and v_{t+1}.

of market integration, that is market structures in which at least some assets can be traded in common by domestic and foreign household. In this interpretation of the model, the absence of international arbitrage is a consequence of one of the households having access to these commonly traded assets.

Subsequently, we remove all assumptions about integration: domestic investors trade domestic assets while foreign investors trade foreign assets. It might seem that such a setting would remove any constraint on the dynamics of the exchange rate. But this is not necessarily the case: we maintain our assumption of the absence of arbitrage opportunities in international markets. Intuitively, this implies that a financial institution having access to both the domestic and foreign asset markets should not be able to earn arbitrage profits. Figure 2C illustrates such a market. This condition often arises in models where international financial trade is operated by financial intermediaries.[14]

3.3 When global shocks play an important role

In this section we consider departures from market completeness where global shocks are important contributors to the variables of interest. We focus on two cases. First, we consider a version of partial market integration where a full set of AD securities is not available, but global shocks explain the entire variation in the depreciation rate. Second, we consider the case of intermediated markets where investors do not trade any common assets, but global shocks explain the entire variation in priced

[14] While their decisions might be affected by various frictions, it is often assumed that they could enter in arbitrage trade. For example, a risk-based constraint such as Value-at-Risk in Basel requirements, does not penalize risk-free trades. Relatedly, theories of limits to arbitrage (Shleifer and Vishny, 1997) often allow arbitrageurs to be unconstrained in engaging in risk-free trades.

risks. The main takeaway is that in such economic scenarios one cannot make much progress towards capturing the currency puzzles.

Integration. Assume that domestic and foreign investors can invest in the risk-free asset of the other country. Lustig and Verdelhan (2019) focus on this setting, which corresponds to $\boldsymbol{r}_{t+1} = \{r_{ft}, r^*_{ft} + \Delta s_{t+1}\}$ and $\boldsymbol{r}^*_{t+1} = \{r^*_{ft}, r_{ft} - \Delta s_{t+1}\}$. Panel A of Figure 4 illustrates this structure.

In this case, shocks to the depreciation rate $\widetilde{\Delta s}_{t+1}$ are spanned by both $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$, and are the only shocks therein. As such, Proposition 1 applies with respect to this shock, which leads to

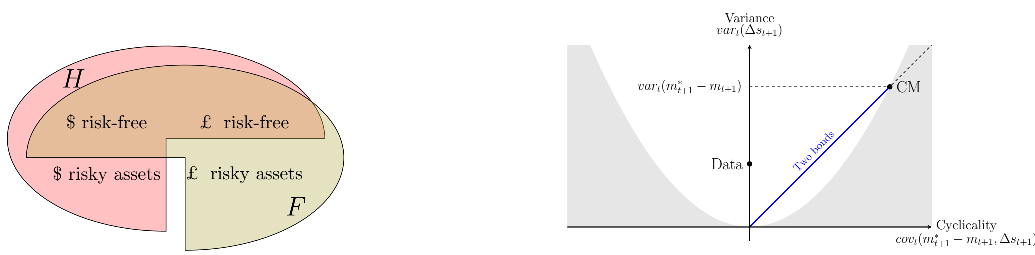
$$\widetilde{\Delta s}_{t+1} = \text{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \widetilde{\Delta s}_{t+1}), \tag{23}$$

because the projection of the depreciation rate on itself is the depreciation rate, global shocks explain 100% of the variation in the depreciation rate. Here $u_{t+1} = 0$ mechanically because the asset spanning the depreciation rate is the depreciation rate itself, or, more precisely, the carry return on the strategy based on risk-free assets. Therefore, the implication of Proposition 2 still coincides with the complete markets case.

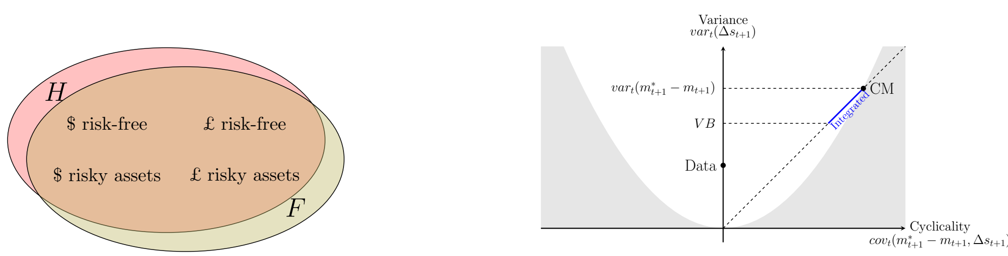
This result is reminiscent of using the difference in minimum-entropy SDFs to infer the depreciation rate in fully integrated markets (e.g., Sandulescu, Trojani, and Vedolin, 2020). In such a setting one necessarily uses the exchange rate to construct the SDFs in each country, and, therefore, the result is the same as Equation (23). However, these asset-based SDFs tell us nothing about how the SDFs appearing in the puzzles, households' intertemporal marginal rates of substitution, relate to the

Figure 4: Volatility and cyclicality when the exchange rate is spanned

A. Two risk-free bonds



B. Fully integrated



The figure illustrates the trade-offs in matching volatility and cyclicality of the exchange rate. The point labeled Data is a stylized representation of the evidence regarding the depreciation rates. The point labeled CM represents the complete market setting where $var_t(\Delta s_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})$. The grey area represents the infeasible combinations of volatility and cyclicality of depreciation rates due to the Cauchy-Schwarz inequality. We consider two scenarios where both domestic and foreign households can trade the risk-free asset in another country. This allows both households to fully span the exchange rate via traded portfolios. The blue lines show the feasible variance-cyclicality combinations in such a scenario. The variance bound, $VB = var_t(\text{proj}(m^*_{t+1} - m_t | \widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1}))$

exchange rate.

Equation (13) with $\theta_t = 0$ and equation (23) are also equivalent to the ones in Proposition 1 of Lustig and Verdelhan (2019). Therefore, we concur with these authors that one can make only limited progress in addressing the three exchange rate puzzles within such a market structure. Specifically,

$$var_t(\Delta s_{t+1}) = var_t(\text{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}|\widetilde{\Delta s}_{t+1})) \leq var_t(m^*_{t+1} - m_{t+1}),$$

which potentially alleviates the volatility puzzle. Next, because the currency risk premium is controlled by exactly the same equation as in the complete-markets case, partial integration with two risk-free bonds does not help in resolving the FX premium puzzle. As regards the cyclicality puzzle, the covariance of the depreciation rate with the SDF differential must equal the variance of the exchange rate,

$$cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) = cov_t(\Delta s_{t+1}, \text{proj}(m^*_{t+1} - m_{t+1}|\widetilde{\Delta s}_{t+1})) = var_t(\Delta s_{t+1}).$$

Therefore, just like in the complete markets case, there is a cyclicality puzzle.[15] The right graph in Figure 4A summarizes these constraints on the cyclicality and volatility of the depreciation rate: the exchange must be on the 45-degree line segment between the origin and the complete markets point.

Next, we show that the restrictions of this setting continue to hold as more assets, either domestic or foreign, are bilaterally traded. Specifically, we allow for a broader set of assets to be traded by both domestic and foreign households. This implies that $H = F = I$ — a structure represented in Panel B of Figure 4. Because these sets include the risk-free bonds, the exchange rate is sill spanned, $u_{t+1} = 0$. Proposition 2 implies the same risk-premium result as in the complete-markets case.

[15] Having said that, the correlation between relative discount factors in the domestic and foreign economies and depreciation rate is less than perfect:

$$corr_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) = \frac{cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})}{\sqrt{var_t(m^*_{t+1} - m_{t+1}) \cdot var_t(\Delta s_{t+1})}} \leq \frac{cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})}{var_t(\Delta s_{t+1})} = 1.$$

In this setting, the domestic household has access to assets with returns $\boldsymbol{r}^*_{t+1} + \Delta s_{t+1}$, where the first element is $r^*_{ft} + \Delta s_{t+1}$. Therefore, this household can trade $\boldsymbol{r}^*_{t+1} - r^*_{ft}$ by going long risky assets and shorting the risk-free asset. Thus, the domestic household can trade the same risks as the foreign one. The household can also isolate currency risk from all other risks by trading this way. The same logic applies to the foreign household's ability to trade domestic risks. Proposition 1 then applies to all traded risks, i.e., $\boldsymbol{\epsilon}^g_{t+1} = (\widetilde{\Delta s}_{t+1}, \widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1})$.

Therefore, the projection of $m^*_{t+1} - m_{t+1}$ on the depreciation rate and asset returns has a loading of one on the depreciation rate and zero on all other assets. Because it is more stringent than the condition with only risk-free assets (equation (23)), this relation still implies tht the cyclicality and variance of the exchange rate coincide. It also adds a lower bound on the volatilty of the exchange rate:

$$
\begin{aligned}
var_t(\Delta s_{t+1}) &= var_t(\text{proj}(m^*_{t+1} - m_t | \Delta s_{t+1}, \widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1})) \\
&\geq var_t(\text{proj}(m^*_{t+1} - m_t | \widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1})).
\end{aligned}
$$

The exchange rate must be more volatile than the projection of the SDF differential on asset returns. As the risky returns span more and more states of the world, this lower bound grows and we converge to the complete markets case. The right panel of Figure 4B illustrates this additional restriction.

Thus, we conclude that bilateral trading in risk-free bonds imposes the critical restrictions on the depreciation rate. That is because trading these bonds amounts to the ability for both households to trade in exchange rate itself, which leads to the projection equation (23).

Intermediated markets. We consider a particular case where all the risks affecting returns and the SDFs are global. This corresponds to:

$$m^*_{t+1} - m_{t+1} = g_{t+1} \tag{24}$$

$$\text{rank}(var_t(\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1})) = \text{rank}(var_t(\boldsymbol{r}_{t+1})) = \text{rank}(var_t(\boldsymbol{r}^*_{t+1})), \tag{25}$$

i.e., there is no $\boldsymbol{\epsilon}_{t+1}$ or $\boldsymbol{\epsilon}^*_{t+1}$. Such a situation would occur in a setting in which the two economies are driven by the same set of shocks, although potentially with different exposure to these shocks. For example, all variation could be driven by a global financial cycle, with the U.S. more sensitive than other countries to this cycle.

In this case, Equation (10) of Proposition 1 simplifies to:

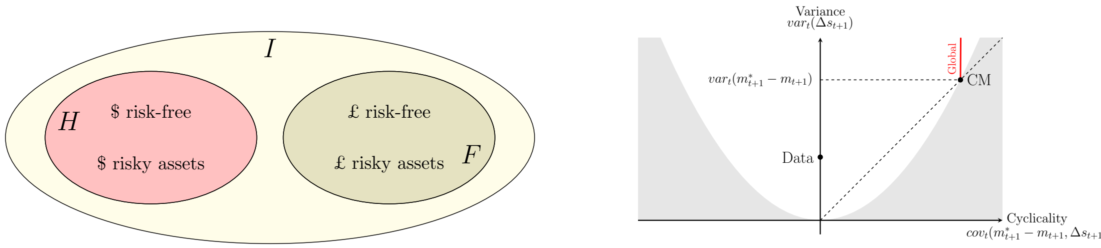
$$\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} = \text{proj}(\widetilde{\Delta s}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}) = g_{t+1}, \quad \ell_{t+1} = 0. \tag{26}$$

The projection of the exchange rate on asset returns is equal to the difference between stochastic discount factors. While this condition is reminiscent of the projection relation with integrated risk-free asset markets, equation (23), the two are different because the projection concerns the depreciation rate instead of the difference of SDFs. Now it is a regression of the exchange rate depreciation on the difference of log SDFs which yields a coefficient of 1. The unspanned component u_{t+1} is unbounded, and Equation (26) implies:

$$var_t(\Delta s_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + var_t(u_{t+1}) \geq var_t(m^*_{t+1} - m_{t+1}).$$

This result deepens the volatility puzzle. If economies are entirely driven by global shocks, exchange rate volatility can only be larger than in the complete market case.

Figure 5: Exchange rate shocks with global shocks and intermediated markets



The figure illustrates the trade-offs in matching volatility and cyclicality of the exchange rate. The point labeled Data is a stylized representation of the evidence regarding the depreciation rates. The point labeled CM represents the complete market setting where $var_t(\Delta s_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})$. The grey area represents the infeasible combinations of volatility and cyclicality of depreciation rates due to the Cauchy-Schwarz inequality. We consider a scenario when markets are intermdiated (as depicted in the left panel) and asset returns are subjected to global shocks only. The red line shows the feasible variance-cyclicality combinations in such a scenario.

The cyclicality is not affected because Equation (26) implies:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) = cov_t(g_{t+1}, g_{t+1} + u_{t+1}) = var(g_{t+1}) = var_t(m^*_{t+1} - m_{t+1}).$$

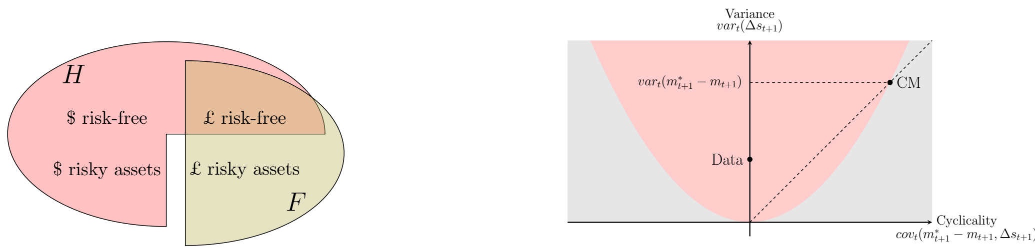
We depict this situation in the right panel of Figure 5 via a vertical line emanating from CM (complete markets case). Global-only risks exacerbate the volatility-cyclicality puzzles associated with the complete markets case.

3.4 When global shocks play a diminished role

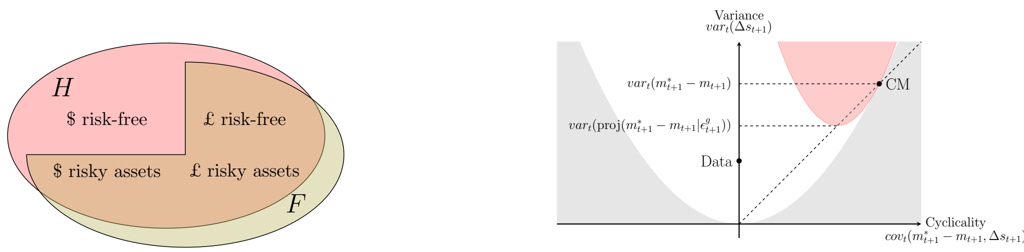
In this section we continue with the partial market integration and intermediated markets settings, but we allow the global shocks to play a smaller role in explaining variation in the variables of interest (the depreciation rate, or priced risks). As we show, such settings result in weaker connection between the SDFs and the depreciation rate, and have better chances of capturing the currency puzzles simultaneously.

Figure 6: Exchange rate shocks across market structures

A. One risk-free bond



B. All but one bond



The figure illustrates the trade-offs in matching volatility and cyclicality of the exchange rate. The point labeled Data is a stylized representation of the evidence regarding the depreciation rates. The point labeled CM represents the complete market setting where $var_t(\Delta s_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})$. The grey area represents the infeasible combinations of volatility and cyclicality of depreciation rates due to the Cauchy-Schwarz inequality. We consider a scenario when markets are partially integrated (as depicted in the left panel). The red line shows the feasible variance-cyclicality combinations in such a scenario. The pink area inside of the cones depicts the feasible combinations of volatility and cyclicality.

Partial integration. First, consider a setting where only the foreign risk-free bond is tradeable by domestic and foreign households. Such a case arises often in the context of sovereign bonds of emerging economies (H), which restrict participation in their market to the investors of their domicile, but these investors are not prevented from trading US bonds (F). This case corresponds to $\boldsymbol{r}_{t+1} = \{r_{ft}, r^*_{ft} + \Delta s_{t+1}\}$ and $\boldsymbol{r}^*_{t+1} = \{r^*_{ft}\}$. The left graph of Figure 6A illustrates this structure.

Proposition 1 requires exposure to a set of common risks, which does not apply in this case. No-arbitrage requirement does not impose any constraints on the shocks to the depreciation rate. Therefore, there is full flexibility to match the volatility and cyclicality puzzles as displayed on the right graph of panel A.

In contrast, Proposition 2 applies precisely because the domestic household has access to the carry trade (based on risk-free bonds). The carry portfolio is the spanning portfolio in this case. Equation (13) implies the same risk premium from the domestic perspective as in complete markets. However, its counterpart with foreign SDF in equation (14) does not hold because the foreign household does not have access to the carry trade. In this setting, the FX risk premium puzzle is unchanged, though only present from the domestic perspective.

Now we extend the setup by allowing domestic and foreign households to trade both domestic and foreign risky assets, see Panel B of Figure 6. The domestic household can trade foreign risks and isolate currency risk from all other risks. The foreign household, however, cannot separate out the currency risk because $r_{ft} - \Delta s_{t+1}$ is inaccessible.

Consider, for example, the case of one risky asset in each country: $\boldsymbol{r}_{t+1} = (r_{ft}, r_{1,t+1}, r^*_{ft} + \Delta s_{t+1}, r^*_{1,t+1} + \Delta s_{t+1})$, $\boldsymbol{r}^*_{t+1} = (r^*_{ft}, r^*_{1t+1}, r_{1,t+1} - \Delta s_{t+1})$. The foreign household can trade the risks in $r^*_{1,t+1}$ and $(r_{1,t+1} - \Delta s_{t+1})$. These risks are accessible to the domestic household as well, so these constitute the set of global shocks. However, there is no trade that can isolate the currency risk for the foreign household. Even though the depreciation rate appears in the construction of global shocks, there is no reason to believe that the projected depreciation rate would be close to the actual one, in general.

With more risky assets, Proposition 1 applies with $\boldsymbol{\epsilon}^g_{t+1} = \widetilde{\boldsymbol{r}}^*_{t+1}$. This leads to a set of tighter restrictions illustrated in Figure 6B, which are, nevertheless looser than the cases considered in Section 3.3. We can apply Equation (22) to obtain the parabola that is qualitatively similar to the ones displayed in Figure 1.

This result highlights the role of which assets are available for trading and reinforces the importance of bilateral trading of risk-free assets for the emergence of the cyclicality and volatility puzzles. Comparing to Figure 4B, it is critical whether both domestic and foreign households can gain exposure to the exchange rate risk. If they can, then the range of feasible combinations of volatility and cyclicality is extremely limited. Furthermore, the fewer risky assets one can trade, the lower is the vertex of the parabola and the higher are the chances to capture the Data point within a model.

Intermediated markets. Consider a case when the two economies are spanned by a distinct set of shocks. While these shocks might be correlated, there is no redundancy between domestic and foreign returns. This corresponds to the condition:

$$\text{rank}(var_t(\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1})) = \text{rank}(var_t(\boldsymbol{r}_{t+1})) + \text{rank}(var_t(\boldsymbol{r}^*_{t+1})). \tag{27}$$

In such a situation, $\boldsymbol{\epsilon}^g_{t+1}$ is empty. Therefore, Proposition 1 applies to an empty set, $g_{t+1} = 0$, and does not constrain the properties of the depreciation rate.

Shocks to the exchange rate, ℓ_{t+1} and u_{t+1}, can have an arbitrary variance and correlation with the asset space. As a result, there is no force in the financial market connecting Δs_{t+1} to $m^*_{t+1} - m_{t+1}$. The cyclicality and volatility puzzles do not have to arise in this setting. The white parabola in the right panel of Figure 5 depicts this scenario.

While condition (27) is about the correlation structure of returns in each of the countries, it connects naturally with the structure of shocks driving the home and foreign economies. To see this, consider the case of the two economies being in autarky on the real side (for example if the two countries consume different goods). Suppose, all shocks to firm productivity and output in each country are driven by vectors of shocks $\boldsymbol{\epsilon}_{t+1}$ and $\boldsymbol{\epsilon}^*_{t+1}$ satisfying $\text{rank}(var_t(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1})) = \text{rank}(var_t(\boldsymbol{\epsilon}_{t+1})) + \text{rank}(var_t(\boldsymbol{\epsilon}^*_{t+1}))$. The lack of global real shocks implies the lack of global financial shocks, i.e., condition (27) holds.

4 Empirical Analysis

In this section we investigate empirically whether a broad collection of asset returns is informative about properties of the exchange rate. We limit the asset set in each country to sovereign bonds and various stock portfolios of that country. There are two interpretations of this choice. First, we ask the empirical question, irrespective of market structure, of how much one can hope to learn about the behavior of exchange rates from knowledge of the price of other assets in their origin currency. Second, we are quantifying the restrictions imposed on the behavior of the exchange rate in economies in which only intermediaries participate in international markets.

We first demonstrate that exchange rates appear to have a large component u_{t+1} unrelated to the returns of other traded assets. Then, we provide methods to characterize global shocks. Both of these exercises lead to the conclusion that, for the data we consider, other assets do not impose strong restrictions on the behavior of exchange rates.

4.1 Data

We consider countries corresponding to G10 currencies between 2/1988 and 12/2022. We consider Germany as the representative country for the euro. Prior to the introduction of the euro, we use the German Deutschemark and splice these series together beginning in 1999. Our analysis focuses at the monthly frequency. We obtain exchange rates from WM/Reuters. Government bond yields are from each country's central bank websites. Monthly bond returns are computed from bond yields using a second-order Taylor approximation. We obtain equity indices from MSCI. For each country, 10 different industry indices and 3 different style equity indices (Large + Mid Cap, Value, Growth) are sourced. Risk-free rates are calculated by dividing the 1-year yield by 12.

4.2 Is the exchange rate spanned?

Motivated by Proposition 2, we ask whether the depreciation rate is spanned by combination of domestic and foreign asset returns. We implement regressions of the form:

$$\Delta s_{t+1} = \alpha + \beta' \boldsymbol{r}_{t+1} + \beta^{*\prime} \boldsymbol{r}^*_{t+1} + u_{t+1}. \tag{28}$$

Here the residual u_{t+1} is a direct estimate of the unspanned component of the depreciation rate in equation (8).

We report the adjusted R^2 of these regressions. Exact spanning corresponds to an R^2 of 1. Furthermore, Proposition 3 highlights that R^2 is an appropriate measure of economic distance to the case of perfect spanning.

Table 1 reports the results. We always report the results for the combination of assets in the United States and another country. Each column in the table corresponds to that other country. Each row reflects a particular combination of assets used in the regression. Broadly speaking, we consider bonds and equities separately and in combination. Within each asset class, we zoom in on various individual contributions.

Table 1: Spanning of depreciation rates by asset returns – R^2

Dependent Variable	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Bonds									
10Y	0.25	0.33	7.49	5.36	4.73	1.05	4.79	4.01	0.92
All Maturities	7.23	7.89	15.72	10.15	13.66	5.67	13.95	11.52	13.65
Stocks									
Mkt	21.67	26.56	6.96	4.44	11.24	16.56	16.20	12.34	12.71
Mkt + Value/Growth	21.60	27.98	6.75	5.06	12.47	17.16	15.91	12.71	13.68
Mkt + Value/Growth + Ind.	35.07	41.61	18.55	22.78	29.41	24.53	24.00	19.61	26.88
Bond + Equity	36.74	45.05	26.79	29.13	36.64	27.95	30.62	25.28	33.80
N	419	395	419	419	406	419	414	419	419

Notes: The table reports the adjusted R^2 of a regression of the depreciation rate on various subsets of asset returns, as in equation (28). Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is a different country's currency relative to the U.S. dollar. The first row uses only 10-year bonds, while the second entertains maturities between 2 and 10 years, obtained from various central banks. The next three row successively add various stock portfolios: the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. The final row considers all assets simultaneously.

Major asset classes do not span exchange rates. When looking at all assets together, the R^2s range from 25% for Switzerland to 45% for Canada (in each case combined with the U.S.). Most of the explanatory power comes from the equity side. For example in the case of Canada, the combination of market, value, growth and industry returns explain 42% of variation in the depreciation rate. While the market alone gets to some substantial amount of variation — 27% for Canada —, the addition of industry returns is particularly informative. Consistent with the evidence in Chernov and Creal (2023), bond returns only explain a modest amount of variation in exchange rates: between 0.2% and 7% for the 10-year bond alone, and between 7.2% and 14% for the combination of bonds at all maturities.

We refer to the observation that asset returns do not span changes in exchange rates as the *financial exchange rate disconnect*. While the R^2s we obtain from regressions on asset returns are meaningfully larger than their counterpart with real quantities, these magnitudes are much too small for leading to meaningful theoretical implications. Taking the strictest definition of absence of arbitrage, only a value of 1 leads to the relevance of Proposition 2. According to Proposition 3, even the largest numbers we measure imply a bound for the expected depreciation that is only $\sqrt{1-0.45} = 67\%$ of the bound with an R^2 of 0, not much tighter. Thus, observing the properties of returns on other assets is not informative about the expected currency depreciation rates.

The flipside of this conclusion is that the unspanned component of the depreciation rates, u_{t+1}, is large. In the context of models of intermediated markets, this result offers more flexibility in capturing realistic currency risk premium. As we discussed in section 3.4, partially integrated markets still imply tight restrictions on the currency premium because Proposition 2 holds.

4.3 Identifying global shocks

In this section we quantify the importance of global shocks ϵ^g_{t+1}, which play the key role in Proposition 1. We do so using two empirical approaches. The undirected approach uses canonical correlation analysis (CCA) to identify these shocks from the asset return data. The directed approach starts from candidates for global shocks such as global macro and financial variables proposed in the literature.

4.3.1 Undirected approach

The CCA procedure finds a US and a foreign portfolio of asset returns consisting of $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$, respectively, such that they have the highest correlation possible in sample. Next, conditional on finding this pair, the procedure looks for the next maximally correlated pair of portfolios that are orthogonal to their first pair. And so on.

According to Definition 1, global shocks would manifest themselves as innovations to portfolios with perfect correlation. In that case, Proposition 1 implies that projections of the depreciation rate and the difference in the SDFs on the global shocks coincide. In the data, even the largest correlation could be less than 1. So, in practice we would have to use an ad-hoc cut-off to decide which portfolios are sufficiently close to each other to constitute a measure of a global shock.

Table 2 reports the results. Each column represents a foreign country. For a given country, each row reports the canonical correlation between the assets of that country and the US assets, reported in order of importance, starting from the largest.

The values of the largest correlations range from 64% for New Zealand to 90% for Canada. In some cases lower ranked correlations are similar to the largest one, like for Canada or the UK. In other cases, the magnitude of correlation drops off quickly, e.g., for New Zealand or Norway. Strictly speaking, the evidence suggests that there are no global shocks amongst the assets that we consider.

Table 2: Maximally correlated shocks across asset markets

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Rank 1	75.27	89.82	83.07	75.01	79.47	64.31	78.33	82.95	85.87
Rank 2	65.00	85.06	74.17	64.43	63.49	53.95	65.72	62.62	78.70
Rank 3	61.16	83.44	66.70	58.71	57.14	41.73	59.57	60.41	73.55
Rank 4	57.04	78.79	64.90	51.31	45.86	35.98	55.55	56.12	68.02
Rank 5	51.01	76.82	52.80	46.81	41.74	31.44	49.63	52.32	65.85
Rank 6	41.67	70.79	44.19	46.62	33.59	25.33	38.94	46.83	62.21
Rank 7	34.19	62.84	42.30	41.94	26.88	22.99	38.20	41.16	55.83
Rank 8	31.57	56.20	36.66	39.57	25.80	14.58	33.82	35.18	51.39
N	419	395	419	419	406	419	414	419	419

*N*otes: The table reports the correlation in % between the maximally correlated portfolios of asset returns between the U.S. and each country. The successive pairs of portfolio are orthogonal to each other, and obtained by canonical correlation analysis. Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is for a different country's assets relative to the U.S. assets. The assets include government bonds of maturities between 2 and 10 years (obtained from various central banks) and various stock portfolios: the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios (from MSCI).

As alluded to earlier, we can be more generous with interpreting the evidence in Table 2 and assign a value of 1 to each estimated correlation that is above a certain threshold. We consider the value of 60% as such a threshold. We denote the matrix of foreign portfolio weights by $\boldsymbol{w}^*$; if there is only one global shock, this is a vector. We ask how much variation in the depreciation rate is explained by global shocks. We implement regressions of the form:

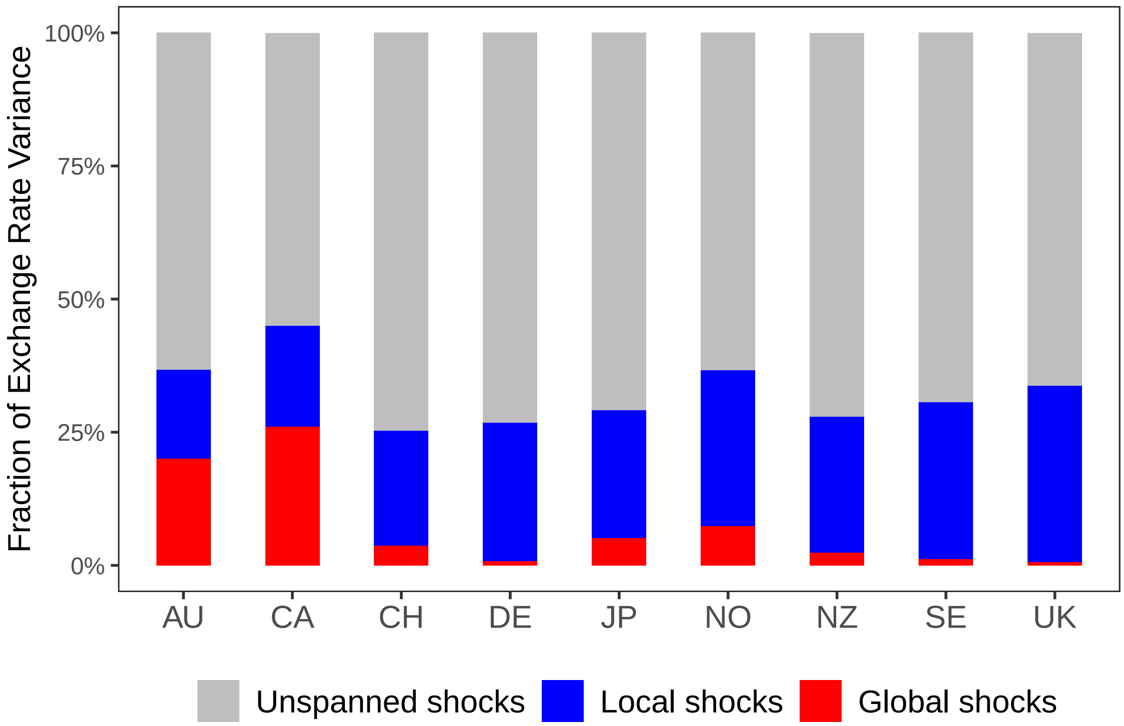
$$\Delta s_{t+1} = \alpha + \beta^{g\prime}(\boldsymbol{w}^{*\prime}\boldsymbol{r}^*_{t+1}) + \varepsilon_{t+1}. \tag{29}$$

The R^2 of such a regression is the fraction variance in exchange rate explained by global shocks. Because we assume that the corresponding domestic portfolio is perfectly correlated with its foreign counterpart, we do not include it in the regression. The regression residual is a direct estimate of the contribution of local and unspanned shocks to the depreciation rate, $\varepsilon_{t+1} = \ell_{t+1} + u_{t+1}$.

Combining with the results of regression (28), we can decompose variation in the depreciation rate into the contribution of global, local, and unspanned shocks. Specifically, we have $var(\beta^{g\prime}(\boldsymbol{w}^{*\prime}\boldsymbol{r}^*_{t+1}))$ for global shocks, and $var(\varepsilon_{t+1}) - var(u_{t+1})$ for local shocks. Figure 7 reports these quantities as fraction of the variation in depreciation rate; the contributions mechanically add up to 1.

For all currencies, at least half of the variation in exchange rates is unspanned by asset returns — the financial disconnect we have already noted. Global shocks contribute very little to variation in the depreciation rates. The contribution is of the order of a few percentage points, with the exception of Australia and Canada with contributions around 25%. These estimates should be seen as an upper bound on the role of global shocks; remember that estimated global shocks include any pair of portfolios with

Figure 7: Decomposition of exchange rate innovations, undirected



*N*otes: The figure reports the fraction of variance in exchange rates explained by globally traded shocks, local shocks, and shocks that are not spanned by asset returns. Each bar is a different country's currency relative to the U.S. dollar. Global shocks are measured via returns of the assets that we use in our analysis using CCA.

correlation above 60%, far from the strict Definition 1.

4.3.2 Directed approach

Instead of being agnostic about the nature of global shocks we rely on macroeconomic research and assume that they are known. Specifically, we take VIX, GFC (Miranda-Agrippino and Rey, 2020), and EBP (Gilchrist and Zakrajsek, 2012) as such shocks.

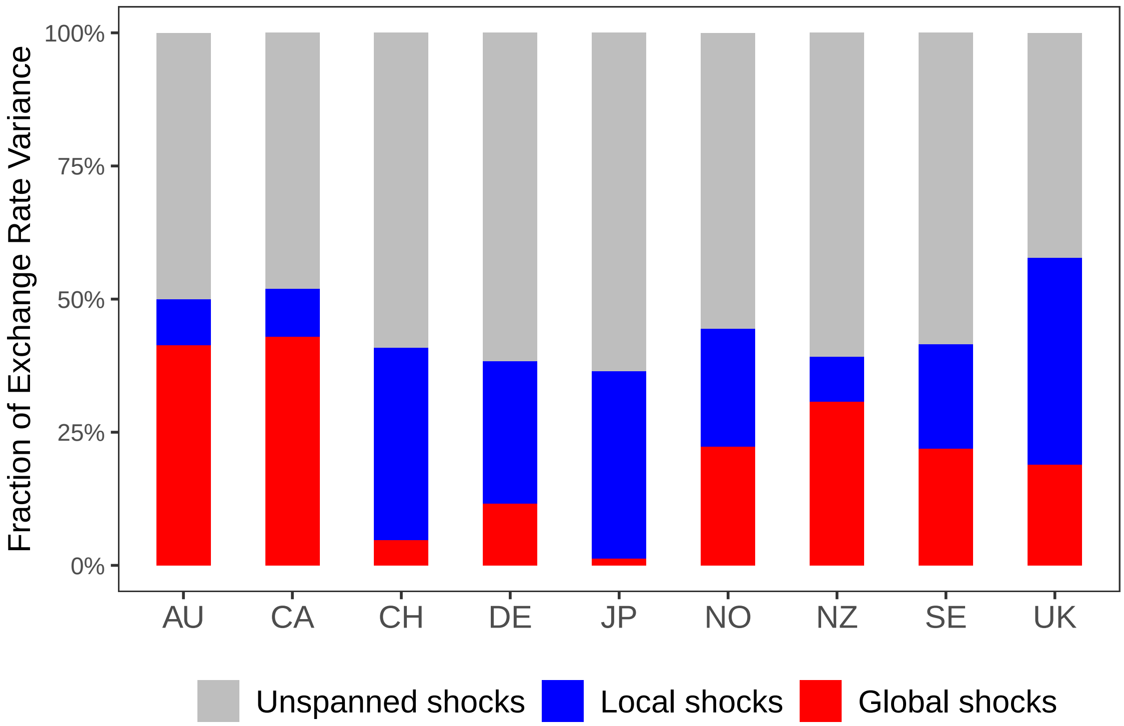
This approach requires a strong assumption that portfolios of traded assets in each economy can span these shocks.

For each country, we regress its depreciation rate vs USD on the global shocks. The R^2 from such a regression produce the fraction of the exchange rate variation due to global shocks. Next, we implement the regression in Equation (28) where the set of returns is complemented by the three global shocks to obtain the unspanned component. Naturally, it is going to be smaller than that in the previous section. The knowledge of the variation due to global and unspanned shocks delivers the variation due to local shocks.

Figure 8 reports the resulting decomposition of the variation in the exchange rate into the three types of shocks. The directed approach delivers somewhat larger contribution of global shocks, but qualitatively the conclusions are unchanged. The unspanned shocks represent the largest share of shocks. Contribution of the gloabl shocks is modest with Australia and Canada, who approach 50%, being the exception.

Just like the financial disconnect leads to weak restrictions about the expected depreciation rate, the small role of global shocks implies weak restrictions about exchange rate risks. The flipside of this conclusion is that the settings of Section 3.4m which have sizable exposure to local shocks relative to global shocks are capable of resolving the cyclicality and volatility puzzles jointly. Given that partially integrated markets still impose tight restrictions on the currency risk premium, the intermediated market structure appears to be the most promising avenue for describing the equilibrium behavior of the exchange rate.

Figure 8: Decomposition of exchange rate innovations, directed



Notes: The figure reports the fraction of variance in exchange rates explained by globally traded shocks, local shocks, and shocks that are not spanned by asset returns. Each bar is a different country's currency relative to the U.S. dollar. Global shocks are measured via VIX, GFC, and EBP.

5 Conclusion

In this paper, we propose a general framework for understanding how much financial markets determine the behavior of exchange rates. Our theory accommodates many settings: complete or incomplete markets, arbitrary forms of market integration, or situations in which international financial trade happens through intermediaries. We characterize restrictions on the behavior of exchange rates due to the absence of international arbitrage. These restrictions can be summarized by two conditions that share the simplicity of the complete market result while having richer implications.

We use these results to study many different market structures. We find that in theoretical settings where financial markets are informative about the exchange rate, they lead to the same counterfactual implications as in complete markets. In contrast, some structures, such as intermediated markets, do not impose much restrictions on exchange rates. This lack of structure is consistent with two properties of the data. First, there is a financial exchange rate disconnect: depreciation rates are not that correlated to asset returns. Second, few shocks are globally traded, and they explain even less of the variation in exchange rates. Thus, we conclude that intermediated market structures are the most promising avenues for modeling the equilibrium exchange rate.

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Appendix

A Identification and construction of the global shocks

We show how to identify a basis for the set of global shocks $\boldsymbol{\epsilon}^g_{t+1}$. We drop time indices for parsimony.

First, recall what canonical correlation analysis does.

Definition 2. *Canonical correlation analysis identifies pairs* $(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}^*_i)$ *for* $i = 1, \ldots, K$ *for some* K *such that:*

1. $\forall i \; var(\boldsymbol{\lambda}'_i \boldsymbol{r}) \neq 0$
2. $\forall i \; \boldsymbol{\lambda}'_i \boldsymbol{r} = \boldsymbol{\lambda}^{*\prime}_i \boldsymbol{r}^*$
3. $\forall i \neq j \; \boldsymbol{\lambda}'_i \boldsymbol{r} \perp \boldsymbol{\lambda}'_j \boldsymbol{r}$
4. $\forall r \in span(\boldsymbol{r}), r^* \in span(\boldsymbol{r}^*)$ *if* $\forall i \; r \perp \boldsymbol{\lambda}'_i \boldsymbol{r}$ *and* $r^* \perp \boldsymbol{\lambda}'_i \boldsymbol{r}$ *then* $r \neq r^*$

Then we show that this procedure identifies a basis of $\boldsymbol{\epsilon}^g$.

Lemma 1. *The collection* $(\boldsymbol{\lambda}'_1 \boldsymbol{r}, \ldots \boldsymbol{\lambda}'_K \boldsymbol{r})$ *identified by canonical correlation analysis is a basis of* $\boldsymbol{\epsilon}^g$.

Proof. By point 2 of Definition 2, all the $\boldsymbol{\lambda}'_i \boldsymbol{r}$ are in $\boldsymbol{\epsilon}^g$. Thus, $span(\boldsymbol{\lambda}'_1 \boldsymbol{r}, \ldots \boldsymbol{\lambda}'_K \boldsymbol{r}) \subset \boldsymbol{\epsilon}^g$.

Let us show the other direction. Assume that there exists $r \in \boldsymbol{\epsilon}^g$ such that $r \notin span(\boldsymbol{\lambda}'_1 \boldsymbol{r}, \ldots \boldsymbol{\lambda}'_K \boldsymbol{r})$. We can orthogonalize r to all the $\boldsymbol{\lambda}'_i \boldsymbol{r}$ and obtain $\hat{r}$. Because $\hat{r}$ is a linear combination of r and $\boldsymbol{\lambda}'_i \boldsymbol{r}$ which are all in $\boldsymbol{\epsilon}^g$, it is also in $\boldsymbol{\epsilon}^g$, and therefore in $span(\boldsymbol{r})$ and $span(\boldsymbol{r}^*)$. By substituting $\hat{r}$ for both r and r^* in point 4 of Definition 2, we immediately obtain a contradiction. Therefore $span(\boldsymbol{\lambda}'_1 \boldsymbol{r}, \ldots \boldsymbol{\lambda}'_K \boldsymbol{r}) \supset \boldsymbol{\epsilon}^g$; the two sets are equal. By point 3 of the CCA definition, $dim(span(\lambda'_1 r, \ldots \lambda'_K r)) = K$, so $(\lambda'_1 r, \ldots \lambda'_K r)$ is indeed a basis of $\boldsymbol{\epsilon}^g$. ■

Furthermore, we relate the dimension of $\boldsymbol{\epsilon}^g$ to the rank of covariance matrices of $\boldsymbol{r}$, $\boldsymbol{r}^*$, and the two combined.

Lemma 2. *The dimension of* $\boldsymbol{\epsilon}^g$ *is:*

$$dim(\boldsymbol{\epsilon}^g) = rank(var(\boldsymbol{r})) + rank(var(\boldsymbol{r}^*)) - rank(var(\boldsymbol{r}, \boldsymbol{r}^*)).$$

Proof. Observe that, by construction,

$$\begin{aligned} dim(span(\boldsymbol{r}, \boldsymbol{r}^*)) &= [dim(span(\boldsymbol{\epsilon}^g)) + dim(span(\boldsymbol{\epsilon}))] + \{dim(span(\boldsymbol{\epsilon}^*))\} \\ &= [dim(span(\boldsymbol{r}))] + \{dim(span(\boldsymbol{r}^*) - dim(\boldsymbol{\epsilon}^g))\}. \end{aligned}$$

Therefore,

$$dim(\boldsymbol{\epsilon}^g) = dim(span(\boldsymbol{r})) + dim(span(\boldsymbol{r}^*)) - dim(span(\boldsymbol{r}, \boldsymbol{r}^*)),$$

which yields the result. ■

B Derivation of the main results

B.1 Portfolio approximation

To maintain tractability, we follow Campbell and Viceira (2002) and approximate the log portfolio excess returns relative to a risk-free rate r_{ft}:

$$\begin{aligned} r_{p,t+1} - r_{ft} &= \log\left(\boldsymbol{w}_t' e^{\boldsymbol{r}_{t+1} - r_{ft}\boldsymbol{\iota}}\right) \\ &\approx \boldsymbol{w}_t'(\boldsymbol{r}_{t+1} - r_{ft}\boldsymbol{\iota}) + \frac{1}{2}\boldsymbol{w}_t' \operatorname{diag}(\boldsymbol{\Sigma}_t) - \frac{1}{2}\boldsymbol{w}_t'\boldsymbol{\Sigma}_t\boldsymbol{w}_t, \end{aligned} \tag{30}$$

where $\boldsymbol{\Sigma}_t$ is the variance-covariance matrix of log returns. This approximation allows us to represent portfolios returns as linear combination of log returns. Importantly, it is stable by recombination, leading to the same result when applied in two steps or all at once for a portfolio of portfolios.

B.2 Two international portfolios

Two international portfolios are useful for the derivation of our main results.

Carry trade. One zero-cost portfolio, often referred to as carry, entails taking long and short positions in related assets:

$$R_{\text{carry},t+1} = R_{t+1} - R^*_{t+1} \cdot S_{t+1}/S_t. \tag{31}$$

Traditionally, the traded assets are taken to be domestic and foreign risk-free (one-period) bonds. But carry does not have to be limited by that. For instance, Lustig, Stathopoulos, and Verdelhan (2013) consider long-term bonds. More generally, one could use any pair of assets that are close to each other, e.g., $corr_t(r_{t+1}, r^*_{t+1}) \approx 1$.

The key characteristic of the carry trade is that it exposes the arbitrageur to currency risk.

Lemma 3. *The conversion from foreign to home returns in the carry portfolio introduces exposure to currency risk,* $\widetilde{r}_{\text{carry},t+1} = \widetilde{r}_{t+1} - \widetilde{r}^*_{t+1} + \widetilde{\Delta s}_{t+1}$.

Proof. We map the zero-cost portfolio (31) into the log approximation of a funded portfolio in equation (30) by adding a position in the risk-free asset:

$$R_{p,t+1} \equiv R_{\text{carry},t+1} + R_{f,t} = R_{t+1} - R^*_{t+1} \cdot S_{t+1}/S_t + R_{f,t}.$$

The portfolio $R_{p,t+1}$ corresponds to the weights $w_1 = 1$ in the domestic risky asset R_{t+1}, $w_2 = -1$ in the foreign risky asset converted to USD, $R^*_{t+1} \cdot S_{t+1}/S_t$, and $w_3 = 1$ in the domestic risk-free asset with $\boldsymbol{w}_t = (w_1, w_2, w_3)'$. These weights lead to an expression for the log gross return relative to the risk-free rate $R_{p,t+1}/R_{f,t}$:

$$\begin{aligned} r_{\text{carry},t+1} &\equiv r_{p,t+1} - r_{ft} \\ &= r_{t+1} - r^*_{t+1} - \Delta s_{t+1} + cov_t(r_{t+1} - r^*_{t+1} - \Delta s_{t+1}, r^*_{t+1} + \Delta s_{t+1}). \end{aligned} \tag{32}$$

Thus, the shocks to the exchange rate have an impact on the portfolio performance. ■

Differential carry. That carry is exposed to currency risk prompts us to consider another zero-cost portfolio, labeled as differential carry, which is long one unit of the domestic asset, and short one unit of the foreign asset, financed at the respective risk-free rates:

$$R_{\text{diff},t+1} = (R_{t+1} - R_{ft}) - (R^*_{t+1} - R^*_{ft}) \cdot S_{t+1}/S_t. \tag{33}$$

Intuitively, this portfolio does not introduce additional currency exposure because, in contrast to carry, only the foreign excess return is converted to USD. We demonstrate this formally in the following lemma.

Lemma 4. *The conversion from foreign to US returns in the diff portfolio does not introduce additional exposure to currency risk,* $\widetilde{r}_{\text{diff},t+1} = \widetilde{r}_{t+1} - \widetilde{r}^*_{t+1}$.

Proof. We map the zero-cost portfolio (33) into a funded portfolio to use the approximation of equation (30):

$$R_{p,t+1} \equiv R_{\text{diff},t+1} + R_{f,t} = R_{t+1} - (R^*_{t+1} - R^*_{ft}) \cdot S_{t+1}/S_t.$$

The portfolio $R_{p,t+1}$ corresponds to the weights $w_1 = 1$ in the domestic risky asset R_{t+1}, $w_2 = -1$ in the foreign risky asset converted to USD, $R^*_{t+1} \cdot S_{t+1}/S_t$, and $w_3 = 1$ in the foreign risk-free asset converted to USD, $R^*_{ft} \cdot S_{t+1}/S_t$, with $\boldsymbol{w}_t = (w_1, w_2, w_3)'$. These weights lead to an expression for the relative log return:

$$\begin{aligned} r_{\text{diff},t+1} &\equiv r_{p,t+1} - r_{ft} \\ &= r_{t+1} - r_{ft} - (r^*_{t+1} - r^*_{ft}) - cov_t(r^*_{t+1}, \Delta s_{t+1}) + cov_t(r^*_{t+1}, r_{t+1} - r^*_{t+1}). \end{aligned} \tag{34}$$

Thus, only the covariance of the foreign return with the exchange rate has a material impact on portfolio performance, not the shocks to the exchange rate. ■

The disappearance of exchange rate risk for the diff returns is in part due to our portfolio approximation. In Appendix Section D, we confirm that this approximation is very tight empirically. We compare the excess returns on various stock portfolios and sovereign bonds in their origin currency and in converted currency. The correlation between the two monthly series is always around 99.9%.

B.3 Proof of Proposition 1

Consider one of the global shocks, ϵ^g_{t+1}. By definition 1, there exist two portfolios $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ such that $\epsilon^g_{t+1} = \widetilde{r}_{p,t+1} = \widetilde{r}^*_{p,t+1}$.

The differential carry portfolio of Lemma 4 is in $\boldsymbol{r}^I_{p,t+1}$. In this case, the portfolio has no risk because $\widetilde{r}_{p,t+1} = \widetilde{r}^*_{p,t+1}$. The shocks to foreign and domestic return perfectly offset each other. By assumption 2, the portfolio must have expected returns equal to the risk-free rate. That is:

$$0 = E_t[r_{p,t+1} - r_{ft}] - E_t[r^*_{p,t+1} - r^*_{ft}] - cov_t(r^*_{p,t+1}, \Delta s_{t+1}) + cov_t(r^*_{p,t+1}, r_{p,t+1} - r^*_{p,t+1}).$$

The last term is equal to 0 because $r_{p,t+1} - r^*_{p,t+1}$ has no risk. We can replace the first two terms by covariances with the SDFs using the domestic and foreign Euler equations (5) and (6),

$$\begin{aligned}0 = &-cov_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2}var_t(r_{p,t+1}) + cov_t(m^*_{t+1}, r^*_{p,t+1}) + \frac{1}{2}var_t(r^*_{p,t+1}) \\ &- cov_t(r^*_{p,t+1}, \Delta s_{t+1}).\end{aligned}$$

Remembering that both portfolio shocks are equal to ϵ^g_{t+1}, this expression simplifies to:

$$cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, \epsilon^g_{t+1}) = 0.$$

This equation is equivalent to

$$cov(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} - \widetilde{\Delta s}_{t+1}, \epsilon^g_{t+1}) = 0,$$

which under log-normality implies equation (10).

Because this result holds for any global shock, it must also hold in terms of multivariate projections on all global shocks $\boldsymbol{\epsilon}^g_{t+1}$. ■

B.4 Proof of Proposition 2

Consider the carry portfolio of Lemma 3 constructed with a pair of portfolios $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ which span the exchange rate (equation (9)). In this case, the portfolio has no risk because $\widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1} = \widetilde{\Delta s}_{t+1}$. The shocks to foreign and domestic return perfectly offset exchange rate risk. By assumption 2, the portfolio must have expected returns equal to the risk-free rate. This corresponds to

$$0 = E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] + cov_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}).$$

The covariance term is equal to 0, because $r_{p,t+1} = r^*_{p,t+1} - \Delta s_{t+1}$ has no risk. We can replace expected returns using the domestic and foreign Euler equations (5) and

(6):

$$E_t[\Delta s_{t+1}] = r_{ft} - cov_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2}var_t(r_{p,t+1})$$
$$- r^*_{ft} + cov_t(m^*_{t+1}, r^*_{p,t+1}) + \frac{1}{2}var_t(r^*_{p,t+1}) = \delta_t$$

We replace $\widetilde{r}_{p,t+1} = \widetilde{r}^*_{p,t+1} + \widetilde{\Delta s}_{t+1}$:

$$\begin{aligned}E_t[\Delta s_{t+1}] &= r_{ft} - r^*_{ft} - cov_t(m_{t+1}, \Delta s_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, r^*_{p,t+1}) \\ &\quad + \frac{1}{2}var_t(r^*_{p,t+1}) - \frac{1}{2}var_t(\Delta s_{t+1}) - \frac{1}{2}var_t(r^*_{p,t+1}) - cov_t(\Delta s_{t+1}, r^*_{p,t+1}) \\ &= r_{ft} - r^*_{ft} - cov_t(m_{t+1}, \Delta s_{t+1}) - \frac{1}{2}var_t(\Delta s_{t+1}) \\ &\quad + cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{p,t+1}).\end{aligned}$$

This proves part b) of Proposition 2. If markets are fully integrated, all asset returns are global shocks, and proposition 1 implies that the last term in the equation above is equal to 0, part a) of the proposition. If the exchange rate is not spanned by asset returns, it is impossible to construct a trade with expected returns involving the expected depreciation rate that is risk-free. Therefore, no arbitrage imposes no restriction on the expected deprectiation rate. ■

B.5 Proof of Proposition 3

Recall our decomposition of the depreciation rate into a spanned and unspanned components, $\Delta s_{t+1} = E_t(\Delta s_{t+1}) + g_{t+1} + \ell_{t+1} + u_{t+1}$. Because $g_{t+1} + \ell_{t+1}$ is spanned by asset returns, there exists $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ such that $\widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1} = g_{t+1} + \ell_{t+1}$. Using Lemma 3, we see that the risk of this portfolio is equal to $var_t(u_{t+1})$. We apply Assumption 3 to relate this risk to the expected return of the carry trade.

$$\left| E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] + cov_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}) + \frac{1}{2}var_t(u_{t+1}) \right|$$
$$\leq B\sqrt{var_t(u_{t+1})}$$

Examining the terms in the left-hand-side, we have:

$$\begin{aligned} E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] &= \delta_t - E_t[\Delta s_{t+1}] = -\psi_t \\ cov_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}) &= cov_t(-u_{t+1}, r_{p,t+1} + u_{t+1}) \\ &= -var_t(u_{t+1}) \end{aligned}$$

Plugging back into the inequality, we obtain:

$$|\psi_t + \frac{1}{2} var_t(u_{t+1})| \leq B\sqrt{var_t(u_{t+1})}.$$

■

C Propositions 1 and 2 are sufficient

We show that the results of Propositions 1 and 2 are not only necessary for the absence of international arbitrage — Assumption 2 — but also sufficient. Specifically we show the following.

Proposition 4. *If:*

1. *Assumption 1 holds,*
2. $E\left(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \epsilon^g_{t+1}\right) = E\left(\widetilde{\Delta s}_{t+1} | \epsilon^g_{t+1}\right)$,
3. (a) *Either* $\exists r^s_{p,t+1} \in \boldsymbol{r}_{p,t+1}, r^{s*}_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ *such that* $\widetilde{\Delta s}_{t+1} = \widetilde{r}^s_{p,t+1} - \widetilde{r}^{s*}_{p,t+1}$ *and*

$$\begin{aligned} E_t\left(\Delta s_{t+1}\right) = r_{f,t} - r^*_{f,t} - cov_t(m^*_{t+1}, \Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) \\ + cov_t\left(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{p,t+1}\right), \end{aligned}$$

   (b) *Or* $\forall r^s_{p,t+1} \in \boldsymbol{r}_{p,t+1}, r^{s*}_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}, \widetilde{\Delta s}_{t+1} \neq \widetilde{r}^s_{p,t+1} - \widetilde{r}^{s*}_{p,t+1}$

then there are no arbitrage opportunities in international markets, Assumption 2 holds.

Proof. We proceed by contradiction. Assume that there exists an international arbitrage:

$$\exists r^I_{p,t+1} \in \boldsymbol{r}^I_{p,t+1}, var_t(r^I_{p,t+1}) = 0 \text{ and } E_t\left(r^I_{p,t+1}\right) \neq r_{ft},$$

and denote $\boldsymbol{w}$ and $\boldsymbol{w}^*$ the set of weights of such a portfolio on $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$. Remember that $1'\boldsymbol{w} + 1'\boldsymbol{w}^* = 1$. We consider the cases of 3a and 3b in turn.

Assume condition 3a holds. As a preliminary, note that this condition is equivalent to saying that a carry portfolio constructed with $r^s_{p,t+1}$ and $r^{s*}_{p,t+1}$ has no risk and no average return in excess of the risk-free rate. Consider the following portfolio: long $\boldsymbol{w}'\boldsymbol{r}_{t+1}$, long $(1'w^*)\, r^s_{p,t+1}$, long $w^{*\prime}\left(r^*_{t+1} + \Delta s_{t+1}\right)$, short $(1'w^*)\, r^{s*}_{p,t+1}$. Because we have added and subtracted the same total weights, the new weights still add up to 1, so this is still a portfolio. Because this portfolio combines two risk-free portfolio, our assumed arbitrage and the risk-free carry trade, its expected return is the sum of the two expected returns, $E_t\left(r^I_{p,t+1}\right)$. The total weight on foreign in the portfolio are $1'\boldsymbol{w}^* - 1'\boldsymbol{w}^* = 0$. Therefore, this trade is a differential carry portfolio. Because it has no risk, its home and foreign leg offset each other. They form a global shock. Applying condition 1 in the proposition and Lemma 4 leads immediately to the result that the portfolio return must equal the risk-free rate. This contradicts the assumption that $E_t\left(r^I_{p,t+1}\right) \neq r_{f,t}$.

Now assume that condition 3b holds. If $1'\boldsymbol{w}^* \neq 0$, then the arbitrage portfolio has a non-zero loading on Δs_{t+1} in addition to the home and foreign returns. Because the portofolio is riskless this implies that we can find a pair of home and foreign returns that spans the depreciation rate, a contradiction of condition 3b. If $1'\boldsymbol{w}^* = 0$, then the two legs of the portfolio in their home currency perfeclty offset each other. Their innovations constitute a global shock and applying condition 1 in the proposition jointly with Lemma 4 implies that the arbitrage portoflio has 0 expected return, a contradiction as well.

D Evaluating the portfolio approximation

We report the correlation (in %) between the excess return on various stock portfolios —Table 3— and bonds of different maturities —Table 5— in their origin currency and converted to U.S. dollars. Tables 4 and 6 start from the U.S. version of these portfolios and converts them to foreign currency. These correlations are pervasively extremely high, almost all over 99.9%.

Table 3: Correlation between excess returns converted in different currencies: foreign stocks

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Market	99.88	99.91	99.93	99.96	99.88	99.89	99.91	99.94	99.94
Value	99.92	99.94	99.93	99.96	99.89	99.85	99.92	99.93	99.94
Growth	99.82	99.88	99.93	99.96	99.9	99.93	99.92	99.95	99.94
Oil, Gas, Coal	99.89	99.93	NA	99.96	99.92	99.92	99.93	NA	99.96
Basic Material	99.84	99.94	99.94	99.95	99.88	99.91	99.91	99.96	99.91
Consumer Discretionary	99.91	99.95	99.93	99.96	99.92	99.94	99.94	99.93	99.96
Consumer Products, Services	99.88	99.96	99.97	99.95	NA	NA	99.94	99.93	99.98
Industrials	99.90	99.91	99.94	99.95	99.89	99.92	99.92	99.94	99.94
Health Care	99.91	99.97	99.96	99.96	NA	99.91	99.93	99.96	99.97
Financials	99.92	99.95	99.94	99.96	99.89	99.93	99.91	99.93	99.92
TeleCom	99.92	99.95	99.96	99.96	99.92	99.84	99.93	99.94	99.96
Technology	99.91	99.88	99.96	99.96	99.86	NA	99.94	99.95	99.95
Utilities	99.93	99.91	99.94	99.97	NA	99.93	NA	99.95	99.97

*N*otes: The table reports the correlation (in %) between the excess return on various stock indices expressed in their home currency and converted to U.S. dollar. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.

Table 4: Correlation between excess returns converted in different currencies: U.S. stocks

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
US Market	99.88	99.94	99.95	99.96	99.87	99.90	99.92	99.94	99.94
US Value	99.90	99.95	99.96	99.96	99.87	99.91	99.92	99.95	99.95
US Growth	99.87	99.93	99.94	99.96	99.88	99.90	99.92	99.94	99.94
US Oil, Gas, Coal	99.90	99.96	99.97	99.98	99.92	99.92	99.94	99.96	99.96
US Basic Material	99.81	99.90	99.92	99.95	99.85	99.88	99.90	99.93	99.93
US Consumer Discretionary	99.91	99.95	99.95	99.96	99.9	99.91	99.92	99.95	99.95
US Consumer Products, Services	99.93	99.97	99.97	99.97	99.92	99.93	99.94	99.96	99.96
US Industrials	99.86	99.93	99.94	99.96	99.84	99.90	99.90	99.94	99.94
US Health Care	99.90	99.96	99.95	99.96	99.88	99.93	99.93	99.95	99.96
US Financials	99.91	99.95	99.95	99.94	99.87	99.93	99.91	99.92	99.94
US TeleCom	99.87	99.93	99.95	99.95	99.9	99.91	99.93	99.96	99.95
US Technology	99.88	99.93	99.94	99.96	99.89	99.91	99.92	99.94	99.94
US Utilities	99.84	99.92	99.94	99.96	99.85	99.88	99.91	99.96	99.94

*N*otes: The table reports the correlation (in %) between the excess return on various stock indices expressed in the U.S. dollars and converted to foreign currency. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.

Table 5: Correlation between excess returns converted in different currencies: foreign bonds

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
2Y Bond	99.86	99.97	99.92	99.97	NA	99.85	99.91	99.91	99.95
3Y Bond	99.86	99.97	99.92	99.97	99.91	NA	NA	99.93	99.96
4Y Bond	NA	99.97	99.93	99.97	NA	NA	NA	99.94	99.96
5Y Bond	99.87	99.97	99.93	99.97	99.91	99.85	99.91	99.93	99.96
6Y Bond	NA	99.96	99.93	99.97	NA	NA	NA	99.92	99.96
7Y Bond	NA	99.96	99.93	99.96	NA	NA	99.91	99.91	99.96
8Y Bond	NA	99.96	99.92	99.96	NA	NA	NA	99.90	99.96
9Y Bond	NA	99.96	99.92	99.96	NA	NA	NA	99.89	99.96
10Y Bond	99.87	99.96	99.93	99.96	99.91	99.88	99.91	99.88	99.96

*N*otes: The table reports the correlation (in %) between the excess return on government bonds of different maturity expressed in their home currency and converted to U.S. dollars. Bond returns are constructed from yields obtained from each country's central bank. Each column corresponds to a different country.

Table 6: Correlation between excess returns converted in different currencies: U.S. bonds

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
US 2Y Bond	99.9	99.95	99.95	99.97	99.91	99.93	99.95	99.93	99.96
US 3Y Bond	99.91	99.96	99.95	99.97	99.92	99.93	99.95	99.92	99.96
US 4Y Bond	99.92	99.96	99.94	99.96	99.92	99.94	99.95	99.91	99.96
US 5Y Bond	99.91	99.97	99.93	99.96	99.91	99.94	99.95	99.89	99.95
US 6Y Bond	99.91	99.97	99.93	99.96	99.89	99.94	99.94	99.88	99.95
US 7Y Bond	99.9	99.96	99.92	99.96	99.88	99.94	99.94	99.86	99.95
US 8Y Bond	99.89	99.96	99.91	99.96	99.86	99.93	99.93	99.85	99.95
US 9Y Bond	99.88	99.96	99.9	99.96	99.85	99.93	99.93	99.84	99.95
US 10Y Bond	99.88	99.96	99.9	99.96	99.84	99.93	99.92	99.83	99.94

Notes: The table reports the correlation (in %) between the excess return on U.S. government bonds of different maturity expressed in U.S.. dollars and converted to foreign currency. Bond returns are constructed from yields obtained from the Federal Reserve. Each column corresponds to a different country.